

# Discrete and continuous dynamic systems

Discrete time linear time-invariant systems: input-output and state space representations

Sampling

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# Lecture overview

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  - CT-LTI system models
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- 3 DT-LTI system models
  - State-space models
  - Pulse response function
  - Discrete difference equation models
  - Pulse transfer operator
- 4 Poles of DT-LTI Systems

# Overview

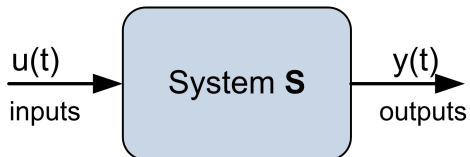
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# Systems

System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

- inputs ( $u$ ) and outputs ( $y$ )



# CT-LTI system models

## Input-output (I/O) models for SISO systems

- time domain
- operator domain

## State-space models

# CT-LTI I/O system models (SISO)

## Transfer function – Linear diff. equation model

$$\begin{aligned}\mathcal{L}\left\{a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y\right\} &= \\ &= \mathcal{L}\left\{b_0 u + b_1 \frac{du}{dt} + \dots + b_m \frac{d^m u}{dt^m}\right\}\end{aligned}$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)}$$

## Transfer function – Impulse response function

$$H(s) = \mathcal{L}\{h(t)\}$$

# CT-LTI state-space models

## General form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) && \text{(state equation)} \\ y(t) &= Cx(t) + Du(t) && \text{(output equation)}\end{aligned}$$

with

- given initial condition  $x(t_0) = x(0)$  and  $x(t) \in \mathcal{R}^n$ ,
- $y(t) \in \mathcal{R}^p$ ,  $u(t) \in \mathcal{R}^r$
- system parameters

$$A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times r}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times r}$$

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1 Previous notions

2 Sampling

- System elements for sampling
- Sampled state-space model

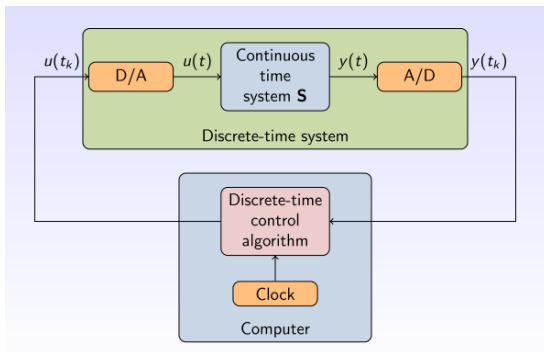
3 DT-LTI system models

4 Poles of DT-LTI Systems



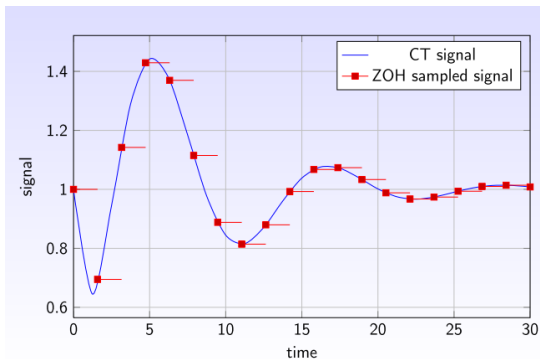
# Sampling

## System elements for sampling



# Zero order hold sampling

## Operation of the D/A converter



# Sampling of CT-LTI systems

**Given:**

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

**Zero order hold sampling of  $u$**

$$u(\tau) = u(t_k) = u(k) \quad , \quad t_k \leq \tau < t_{k+1}$$

**Equidistant (periodic) sampling:**  $t_{k+1} - t_k = h = \text{const}$

**Compute:**

the state-space model of the sampled (discrete time) system

# Sampled state equations - 1

Use the solution of the continuous time state equation

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau \quad (*)$$

Substitute  $t = t_{k+1}$  and  $t_0 = t_k$  with *periodic sampling* ( $h = (t_{k+1} - t_k)$ ) and  $\theta = \tau - t_k$ .

With  $x(k) = x(t_k)$  and  $x(k+1) = x(t_{k+1})$  we obtain from (\*)

$$x(k+1) = e^{Ah}x(k) + e^{Ah} \int_0^h e^{-A\theta}d\theta Bu(k)$$

**Discrete time state equation**

$$x(k+1) = e^{Ah}x(k) + A^{-1}(e^{Ah} - I)Bu(k)$$

# Matrix functions

Given a univariate real function  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  with a square matrix  $A \in \mathbb{R}^{n \times n}$ . Then  $\varphi(A)$  is a square matrix  $\varphi(A) \in \mathbb{R}^{n \times n}$ .

## Matrix exponential function

Given  $A \in \mathbb{R}^{n \times n}$  and the real-valued exponential function  $e : \mathbb{R} \mapsto \mathbb{R}$

Take the Taylor-series expansion of  $e$  around  $t = 0$

$$e^t = 1 + t + \frac{1}{2}t^2 + \dots + \frac{1}{j!}t^j + \dots$$

Substitute  $t = A$  and  $1 = I$

$$e^A = I + A + \frac{1}{2}A^2 + \dots + \frac{1}{j!}A^j + \dots \in \mathbb{R}^{n \times n}$$

For any **diagonal matrix**  $\Lambda$  the matrix function  $\varphi(\Lambda)$  is easy to compute

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \lambda_n \end{bmatrix}, \quad \varphi(\Lambda) = \begin{bmatrix} \varphi(\lambda_1) & 0 & \dots & 0 \\ 0 & \varphi(\lambda_2) & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \varphi(\lambda_n) \end{bmatrix}$$

# Sampled state equations - 2

## Discrete time state equation

$$x(k+1) = e^{Ah}x(k) + A^{-1}(e^{Ah} - I)Bu(k)$$

## DT-LTI state equation for sampled systems

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

with

$$\Phi = e^{Ah} = I + Ah + \dots, \quad \Gamma = A^{-1}(e^{Ah} - I)B = \left( Ah + \frac{Ah^2}{2!} + \dots \right) B$$

# DT-LTI state-space models

$$\begin{aligned}x(k+1) &= \Phi x(k) + \Gamma u(k) && \text{(state equation)} \\y(k) &= Cx(k) + Du(k) && \text{(output equation)}\end{aligned}$$

with given initial condition  $x(0)$  and

$$x(k) \in \mathbb{R}^n, \quad y(k) \in \mathbb{R}^p, \quad u(k) \in \mathbb{R}^r$$

being vectors of finite dimensional spaces and

$$\Phi \in \mathbb{R}^{n \times n}, \quad \Gamma \in \mathbb{R}^{n \times r}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times r}$$

being matrices

# Solution of the DT-LTI state equation

$$x(1) = \Phi x(0) + \Gamma u(0)$$

$$x(2) = \Phi x(1) + \Gamma u(1) = \Phi^2 x(0) + \Phi \Gamma u(0) + \Gamma u(1)$$

$$x(3) = \Phi x(2) + \Gamma u(2) = \Phi^3 x(0) + \Phi^2 \Gamma u(0) + \Phi \Gamma u(1) + \Gamma u(2)$$

..

..

$$x(k) = \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1} \Gamma u(j)$$



# Discrete time signals

$$u = \{u(k), k = 0, 1, \dots\}$$

*scalar valued discrete time signal:  $u(k) \in \mathbb{R}$*

**Pulse signal (scalar valued):** the discrete time analogue for the Dirac-delta (unit impulse) signal

$$u(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

## DT-LTI SISO I/O system models – Pulse response function

From the solution of the state equation with  $D = 0$  and  $x(0) = 0$

$$\begin{aligned}x(k) &= \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1} \Gamma u(j) \\y(k) &= Cx(k) = C\Phi^k x(0) + \sum_{j=0}^{k-1} C\Phi^{k-j-1} \Gamma u(j)\end{aligned}$$

### Pulse response function

$$h(k) = \begin{cases} 0 & k < 1 \\ C\Phi^{k-1}\Gamma & k \geq 1 \end{cases}$$

The *discrete time analogue of the impulse response function*.

# Transformation of the states

Consider the DT-LTI state-space model

$$x(k+1) = \Phi x(k) + \Gamma u(k) \quad , \quad y(k) = Cx(k) + Du(k)$$

with the state transformation  $\bar{x} = Tx$ .

The parameters of the transformed model (another equivalent realization)

$$\bar{\Phi} = T\Phi T^{-1} \quad , \quad \bar{\Gamma} = T\Gamma \quad , \quad \bar{C} = CT^{-1}$$

**Discrete time Markov parameters:**  $C\Phi^{k-1}\Gamma$

- they are invariant for the state transformations

# Shift operators

Definition (forward shift operator  $q$ )

which acts on a discrete time signal as follows

$$qf(k) = f(k + 1) \quad (1)$$

Definition (backward shift operator (delay)  $q^{-1}$ )

which acts on a discrete time signal as follows

$$q^{-1}f(k) = f(k - 1) \quad (2)$$

- **The induced norm of an operator  $q$**  on the vector space  $X$  induced by a norm  $\|\cdot\|$  on the same space is defined as

$$\|q\| = \sup_{\|x\|=1} \frac{\|q(x)\|}{\|x\|}$$

# DT-LTI SISO I/O system models – Discrete difference equation models

- **Forward difference form** with  $n_a \geq n_b$  (proper)

$$y(k+n_a) + a_1 y(k+n_a-1) + \dots + a_{n_a} y(k) = b_0 u(k+n_b) + \dots + b_{n_b} u(k)$$

$$A(q)y(k) = B(q)u(k)$$

$$A(q) = q^{n_a} + a_1 q^{n_a-1} + \dots + a_{n_a}, \quad B(q) = b_0 q^{n_b} + b_1 q^{n_b-1} + \dots + b_{n_b}$$

- **Backward difference form** where  $d = n_a - n_b > 0$  is the *pole excess* (time delay)

$$y(k) + a_1 y(k-1) + \dots + a_{n_a} y(k-n_a) = b_0 u(k-d) + \dots + b_{n_b} u(k-d-n_b)$$

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d),$$

$$A^*(q^{-1}) = q^{n_a} A(q^{-1}), \quad B^*(q^{-1}) = q^{n_b} B(q^{-1})$$

## DT-LTI SISO I/O system models – Pulse transfer operator

- Computed from the DT-LTI state-space model

$$x(k+1) = \Phi x(k) + \Gamma u(k) \quad , \quad y(k) = Cx(k) + Du(k)$$

$$x(k+1) = qx(k) = \Phi x(k) + \Gamma u(k)$$

$$x(k) = (qI - \Phi)^{-1} \Gamma u(k)$$

$$y(k) = Cx(k) + Du(k) = [C(qI - \Phi)^{-1} \Gamma + D]u(k)$$

**Pulse-transfer operator**  $H(q)$  of the SSR  $(\Phi, \Gamma, C, D)$ :

$$H(q) = C(qI - \Phi)^{-1} \Gamma + D$$

The *discrete time analogue of the transfer function*.

It is also *invariant for the state transformation*.

## DT-LTI SISO I/O system models – Pulse transfer operator

- For SISO LTI systems  $H(q)$  is a rational function

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)}, \quad \deg B(q) < \deg A(q) = n$$

where  $A(q)$  is the characteristic polynomial of the state matrix  $\Phi$ .

- Relation with the **discrete difference equation form**

$$\begin{aligned} y(k + n_a) + a_1 y(k + n_a - 1) + \dots + a_{n_a} y(k) &= \\ &= b_0 u(k + n_b) + \dots + b_{n_b} u(k) \end{aligned}$$

$$A(q)y(k) = B(q)u(k)$$

## Poles of DT-LTI systems – 1

- Comparison

	continuous time system	discrete time system
state eq.	$\dot{x}(t) = Ax(t) + Bu(t)$	$x(k+1) = \Phi x(k) + \Gamma u(k)$ $\Phi = e^{Ah}$
output eq.	$y(t) = Cx(t)$	$y(k) = Cx(k)$
poles	$\lambda_i(A)$	$\lambda_i(\Phi)$ $\lambda_i(\Phi) = e^{\lambda_i(A)h}$



## Poles of DT-LTI systems – 2

