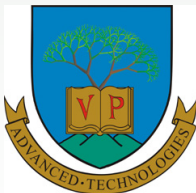


# Discrete and Continuous Dynamical Systems

Attila Magyar

University of Pannonia  
Faculty of Information Technology  
Department of Electrical Engineering and Information Systems



Discrete and continuous dynamical systems:  
Minimal realizations

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# Overview

- 1 Basic notions
- 2 Realizations in special form
  - Controllable canonical form
  - Observable canonical form
  - Diagonal form
- 3 Joint controllability and observability
- 4 General decomposition theorem

# Transformation of states

Two different state space models with the same input-output behavior

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) & , & & \dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) & , & & \mathbf{y}(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{\mathbf{D}}\mathbf{u}(t)\end{aligned}$$

which are related by the transformation

$$\mathbf{T} \in \mathbb{R}^{n \times n} \quad , \quad \det \mathbf{T} \neq 0 \quad , \quad \bar{\mathbf{x}} = \mathbf{T}\mathbf{x} \quad \Rightarrow \quad \mathbf{x} = \mathbf{T}^{-1}\bar{\mathbf{x}}$$

$$\dim \mathcal{X} = \dim \bar{\mathcal{X}} = n$$

$$\mathbf{T}^{-1}\dot{\bar{\mathbf{x}}} = \mathbf{A}\mathbf{T}^{-1}\bar{\mathbf{x}} + \mathbf{B}\mathbf{u}$$

$$\dot{\bar{\mathbf{x}}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\bar{\mathbf{x}} + \mathbf{T}\mathbf{B}\mathbf{u} \quad , \quad \mathbf{y} = \mathbf{C}\mathbf{T}^{-1}\bar{\mathbf{x}} + \mathbf{D}\mathbf{u}$$

$$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \quad , \quad \bar{\mathbf{B}} = \mathbf{T}\mathbf{B} \quad , \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \quad , \quad \bar{\mathbf{D}} = \mathbf{D}$$

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# Controllable canonical form (controller form)

- $H(s) = \frac{b(s)}{a(s)}$
- **Controllability canonical form** of the state space model

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -a_1 & \dots & -a_{n-1} & -a_n \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [ b_1 \quad b_2 \quad \dots \quad b_n ] \mathbf{x}(t)$$

- The change of the  $i$ -th state variable depends on the  $i - 1$ -th one,  $i > 1$
- The change of  $x_1$  depends on all states and the input
- **Always controllable**

# Observable canonical form

- $H(s) = \frac{b(s)}{a(s)}$
- **Observability canonical form** of the state space model

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0 \ \dots \ 0] \mathbf{x}(t)$$

- Each state variable is fed back to the previous one and the output of the system is  $x_1$ .
- **Always observable**

# Diagonal form (or modal form) realization

- State space model in diagonal form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}_D \mathbf{x}(t) + \mathbf{B}_D u(t) \\ y(t) &= \mathbf{C}_D \mathbf{x}(t)\end{aligned}$$

with

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} \lambda_1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \lambda_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} u \\ y &= \begin{bmatrix} c_1 & \cdot & \cdot & \cdot & c_n \end{bmatrix} \mathbf{x}\end{aligned}$$



## Controllability in diagonal form realization

- Controllability matrix

$$\begin{aligned}
 C_n &= [ B \quad AB \quad \dots \quad A^{n-1}B ] = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \dots & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & \dots & \dots \end{bmatrix} = \\
 &= \begin{bmatrix} b_1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \dots & \dots & \lambda_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \lambda_n & \dots & \dots & \lambda_n^{n-1} \end{bmatrix}
 \end{aligned}$$

- The last matrix is a Vandermonde matrix  $V$  with determinant

$$\det V = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$$

- Full rank of the controllability matrix

$$\text{rank } C_n = n \Leftrightarrow \det C_n = \prod_i b_i \prod_{j < i} (\lambda_i - \lambda_j) \neq 0$$

# Controllability and observability in diagonal form realization

## Theorem (Controllability)

*DSSR is controllable iff  $\lambda_i \neq \lambda_j, (i \neq j)$  and  $b_i \neq 0, \forall i$*

## Theorem (Observability)

*DSSR is observable iff  $\lambda_i \neq \lambda_j, (i \neq j)$  and  $c_i \neq 0, \forall i$*

## The transfer function of diagonal form realization

- Transfer function

$$H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \sum_{i=1}^n \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)}$$

where  $\mathbf{I}$  is a unit matrix.

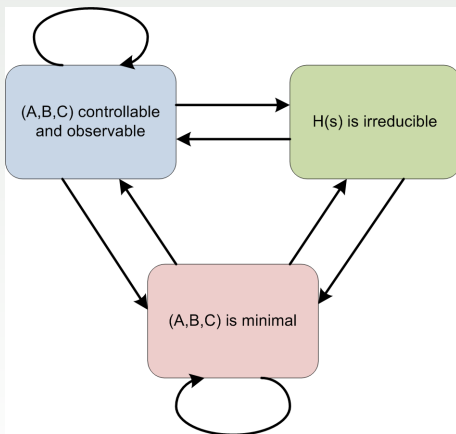
- If either  $c_j = 0$  or  $b_k = 0$  then the transfer function can be described by smaller number of partial fractions than the original:

$$H(s) = \sum_{i=1}^{\bar{n}} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)} \quad , \quad \bar{n} < n$$

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# Equivalent SSR properties



# Overview

Consider **SISO CT-LTI** systems with realization  $(A, B, C)$

- Joint controllability and observability is a **system property**
- Equivalent necessary and sufficient conditions
- Minimality of SSRs
- Irreducibility of the transfer function

## Hankel matrices

- **Definition** A Hankel matrix is a block matrix of the following form

$$\mathbf{H}[1, n-1] = \begin{bmatrix} \mathbf{CB} & \mathbf{CAB} & \dots & \dots & \mathbf{CA}^{n-1}\mathbf{B} \\ \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \dots & \dots & \mathbf{CA}^n\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{CA}^{n-1}\mathbf{B} & \mathbf{CA}^n\mathbf{B} & \dots & \dots & \mathbf{CA}^{2n-2}\mathbf{B} \end{bmatrix}$$

- It contains *Markov parameters*  $\mathbf{CA}^i\mathbf{B}$  that are invariant under state transformations.

## Lemma 1

## Lemma (1)

If we have a system with transfer function  $H(s) = \frac{b(s)}{a(s)}$  and there is an  $n$ -th order realization  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ , which is controllable and observable then all other  $n$ -th order realizations are controllable and observable.

*Proof*

$$\mathcal{O}(\mathbf{C}, \mathbf{A}) = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}, \quad \mathcal{C}(\mathbf{A}, \mathbf{B}) = [ \mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B} ]$$

$$\mathbf{H}[1, n-1] = \mathcal{O}(\mathbf{C}, \mathbf{A})\mathcal{C}(\mathbf{A}, \mathbf{B})$$



# Definitions

## Definition (Relative prime polynomials)

Two polynomials  $a(s)$  and  $b(s)$  are *coprime* (or relative primes) iff  $a(s) = \prod (s - \alpha_i)$ ;  $b(s) = \prod (s - \beta_j)$  and  $\alpha_i \neq \beta_j$  for all  $i, j$ . In other words: the polynomials have no common factors.

## Definition (Irreducible transfer function)

A transfer function  $H(s) = \frac{b(s)}{a(s)}$  is called to be irreducible if the polynomials  $a(s)$  and  $b(s)$  are relative primes.

## Lemma 2

## Lemma (2)

*If we have a controller form realization which is jointly controllable and observable then  $a(s)$  and  $b(s)$  are relative primes ( $H(s)$  is irreducible).*

*Proof*

- A controller form realization is controllable and

$$\mathcal{O}_c = \tilde{\mathbf{I}}_n b(\mathbf{A}_c)$$

$$\tilde{\mathbf{I}}_n = \begin{bmatrix} 0 & \cdot & \cdot & 1 \\ 0 & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

- Non-singularity of  $b(\mathbf{A}_c)$

## Proof of Lemma 2

$$\tilde{\mathbf{I}}_n = [ \mathbf{e}_n \quad \mathbf{e}_{n-1} \quad \cdot \quad \cdot \quad \mathbf{e}_1 ] = \begin{bmatrix} \mathbf{e}_n^T \\ \mathbf{e}_{n-1}^T \\ \cdot \\ \cdot \\ \mathbf{e}_1^T \end{bmatrix}, \quad \mathbf{e}_i = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \end{bmatrix} \leftarrow i.$$

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & \cdot & \cdot & \cdot & -a_n \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{bmatrix}, \quad \mathbf{e}_i^T \mathbf{A}_c = \begin{cases} [-a_1 & -a_2 & \dots & -a_n] \\ \mathbf{e}_{i-1}^T \end{cases}$$

# Proof of Lemma 2

- Computation of the observability matrix  $\mathcal{O}_c = \tilde{I}_n b(\mathbf{A}_c) \in \mathbb{R}^{n \times n}$
- 1st row:

$$\mathbf{e}_n^T b(\mathbf{A}_c) = \mathbf{e}_n^T b_1 \mathbf{A}_c^{n-1} + \dots + \mathbf{e}_n^T b_{n-1} \mathbf{A}_c + \mathbf{e}_n^T b_n \mathbf{I}_n$$

$$n\text{-th term: } [0 \ \dots \ 0 \ b_n]$$

$$(n-1)\text{-th term: } b_{n-1} \mathbf{e}_n^T \mathbf{A}_c = b_{n-1} \mathbf{e}_{n-1}^T = [0 \ \dots \ b_{n-1} \ 0]$$

...

$$\mathbf{e}_n^T b(\mathbf{A}_c) = [b_1 \ \dots \ b_{n-1} \ b_n] = \mathbf{C}_c$$

- 2nd row:

$$\mathbf{e}_{n-1}^T b(\mathbf{A}_c) = \mathbf{e}_{n-1}^T \mathbf{A}_c b(\mathbf{A}_c) = \mathbf{e}_{n-1}^T b(\mathbf{A}_c) \mathbf{A}_c \Rightarrow \mathbf{e}_{n-1}^T b(\mathbf{A}_c) = \mathbf{C}_c \mathbf{A}_c$$

- and so on ...

# Proof of Lemma 2

$\mathcal{O}_c$  is nonsingular

- iff  $b(\mathbf{A}_c)$  is nonsingular because matrix  $\tilde{\mathbf{I}}_n$  is always nonsingular
- $b(\mathbf{A}_c)$  is nonsingular iff  $\det(b(\mathbf{A}_c)) \neq 0$   
which depends on the eigenvalues of  $b(\mathbf{A}_c)$  matrix
- the eigenvalues of the matrix  $b(\mathbf{A}_c)$  are  $b(\lambda_i)$ ,  $i = 1, 2, \dots, n$   
 $\lambda_i$  is an eigenvalue of  $\mathbf{A}_c$ , i.e. a root of  $a(s) = \det(s\mathbf{I} - \mathbf{A})$

$$\det(b(\mathbf{A}_c)) = \prod_{i=1}^n b(\lambda_i) \neq 0$$

$\Updownarrow$   $a(s)$  and  $b(s)$  have no common roots, i.e. they are relative primes

# Minimal realization conditions

## Theorem (1)

$H(s) = \frac{b(s)}{a(s)}$  is irreducible iff all  $n$ -th order realizations are jointly controllable and observable.

*Proof:* combine Lemma 1. and 2.

## Definition (Minimal realization)

A realization  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$  of dimension  $n$  is minimal if one cannot find another realization of dimension less than  $n$ .

## Theorem (2)

$H(s) = \frac{b(s)}{a(s)}$  is irreducible iff any of its realization  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$  is minimal where  $H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$

*Proof:* by contradiction

# Minimal realization conditions

## Theorem (3)

*A realization  $(A, B, C)$  is minimal iff the system is jointly controllable and observable.*

*Proof:* Combine Theorem 1 and Theorem 2 .

## Lemma (3)

*Any two minimal realizations can be connected by a unique similarity transformation (which is invertible).*

*Proof:* (Just the idea of it)

$$T = \mathcal{O}^{-1}(C_1, A_1)\mathcal{O}(C_2, A_2) = \mathcal{C}(A_1, B_1)C^{-1}(A_2, B_2)$$

exists and it is invertible: this is used as a transformation matrix.

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# General decomposition theorem

Given an  $(A, B, C)$  SSR, it is always possible to transform it to another realization  $(\bar{A}, \bar{B}, \bar{C})$  with partitioned state vector and matrices

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_{co} & \bar{\mathbf{x}}_{c\bar{o}} & \bar{\mathbf{x}}_{\bar{c}o} & \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix}^T$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix}$$

# General decomposition theorem

The partitioning defines **subsystems**

- *Controllable and observable subsystem*:  $(\bar{\mathbf{A}}_{co}, \bar{\mathbf{B}}_{co}, \bar{\mathbf{C}}_{co})$  is minimal, i.e.  $\bar{n} \leq n$  and

$$H(s) = \bar{\mathbf{C}}_{co}(s\mathbf{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{B}}_{co} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

- *Controllable subsystem*

$$\left( \left[ \begin{array}{cc} \bar{\mathbf{A}}_{co} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} \end{array} \right], \left[ \begin{array}{c} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \end{array} \right], \left[ \bar{\mathbf{C}}_{co} \quad \mathbf{0} \right] \right)$$

- *Observable subsystem*

$$\left( \left[ \begin{array}{cc} \bar{\mathbf{A}}_{co} & \bar{\mathbf{A}}_{13} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} \end{array} \right], \left[ \begin{array}{c} \bar{\mathbf{B}}_{co} \\ \mathbf{0} \end{array} \right], \left[ \bar{\mathbf{C}}_{co} \quad \bar{\mathbf{C}}_{\bar{c}o} \right] \right)$$

- *Uncontrollable and unobservable subsystem*

$$\left( \left[ \bar{\mathbf{A}}_{\bar{c}\bar{o}} \right], \mathbf{0}, \mathbf{0} \right)$$