Discrete and Continuous Dynamical Systems

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Discrete and continuous dynamical systems:
Observability and controllability

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Overview

CT-LTI state-space models

- General form - revisited

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \quad , \quad \boldsymbol{x}(t_0) = \boldsymbol{x_0} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\end{aligned}$$

with

- signals: $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{y}(t) \in \mathbb{R}^p$, $\boldsymbol{u}(t) \in \mathbb{R}^r$
- system parameters: $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times r}$, $\boldsymbol{C} \in \mathbb{R}^{p \times n}$ $(D = 0)$

DT-LTI state-space models

General form

$$\boldsymbol{x}(k+1) = \Phi \boldsymbol{x}(k) + \Gamma \boldsymbol{u}(k) \qquad \text{(state equation)}$$
$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \qquad \text{(output equation)}$$

with

- given initial condition $\boldsymbol{x}(0) = \boldsymbol{x_0}$,
- $\boldsymbol{x}(k) \in \mathbb{R}^n, \quad \boldsymbol{y}(k) \in \mathbb{R}^p, \quad \boldsymbol{u}(k) \in \mathbb{R}^r$
- system parameters

$$\Phi \in \mathbb{R}^{n\times n} \ , \ \Gamma \in \mathbb{R}^{n\times r} \ , \ \boldsymbol{C} \in \mathbb{R}^{p\times n} \ , \ \boldsymbol{D} \in \mathbb{R}^{p\times r}$$

Overview

Controllability of CT-LTI systems

- Problem statement
  - *Given*:
    - a state-space model with parameters $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$
    - an initial state $\boldsymbol{x}(t_1)$ and a final state $\boldsymbol{x}(t_2) \neq \boldsymbol{x}(t_1)$
  - *Compute*:
    an input signal $\boldsymbol{u}(t)$ which moves the system from $\boldsymbol{x}(t_1)$ to $\boldsymbol{x}(t_2)$ in finite time

Controllability of CT-LTI systems

Theorem (Controllability)

Given $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ *for*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\end{aligned}$$

This SSR with state space $\mathcal{X}$ *is* state controllable ***iff*** *the* controllability matrix $\mathcal{C}_n$ *is of* ***full rank***

$$\mathcal{C}_n = \left[\begin{array}{cccccc} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} & . & . & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{array}\right]$$

Kalman rank condition: If $dim\mathcal{X} = n$ *then* $rank\ \mathcal{C}_n = n$.

- Necessary and sufficient condition

Controllability of CT-LTI systems

Proof.

- Apply the Dirac-delta (Single Input case) function as input to the system, i.e. $u(t) = \delta(t)$ with $\boldsymbol{C} = \boldsymbol{I}$

$$\boldsymbol{x}(t) = h(t) = \boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{B} \quad , \quad \boldsymbol{y}(t) = \boldsymbol{x}(t) \quad , \quad \boldsymbol{x}(0_-) = h(0_-) = \boldsymbol{B}$$

- Then with $\dot{h}(t) = \boldsymbol{A}h(t)$

$$\begin{array}{rcl} \mathbf{S}[u(t) = \delta(t)] & = & h(t) \\ \mathbf{S}[u(t) = \dot{\delta}(t)] & = & \dot{h}(t) = \boldsymbol{A}h(t) \\ \mathbf{S}[u(t) = \ddot{\delta}(t)] & = & \ddot{h}(t) = \boldsymbol{A}^2 h(t) \\ & \vdots & \end{array}$$

- Assume the **input**: $u(t) = g_1\delta(t) + g_2\dot{\delta}(t) + \ldots + g_n\delta^{(n-1)}(t)$
- The superposition principle gives:

$$\begin{array}{l} x(0_+) = x(0_-) + g_1 h(0_-) + g_2\dot{h}(0_-) + \ldots + g_n h^{(n-1)}(0_-) \\ x(0_+) = x(0_-) + g_1\boldsymbol{B} + g_2\boldsymbol{A}\boldsymbol{B} + \ldots + g_n\boldsymbol{A}^{n-1}\boldsymbol{B} \end{array}$$

$\square$

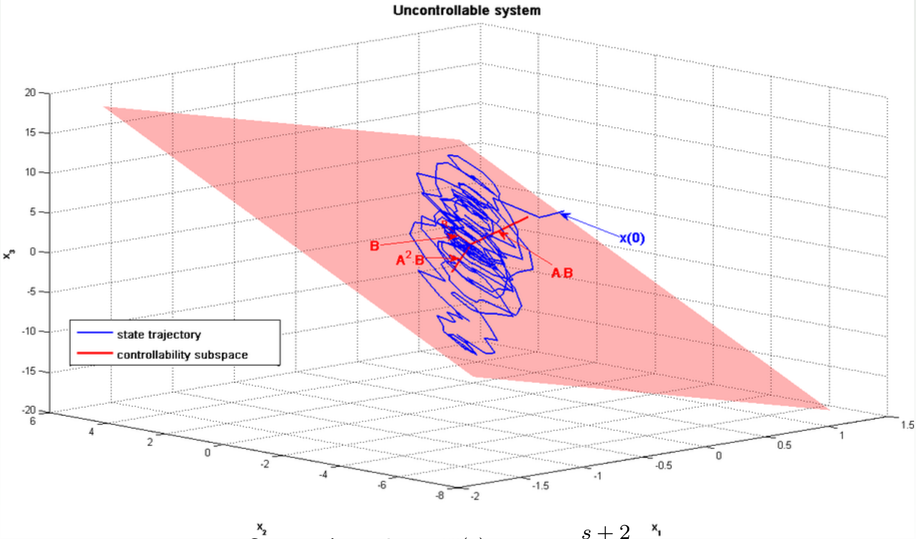
Controllability of CT-LTI systems

- Assuming $\boldsymbol{x}(0_-) = 0$ we get

$$\boldsymbol{x}(0_+) = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} & . & . & . & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ . \\ . \\ . \\ g_n \end{bmatrix}$$

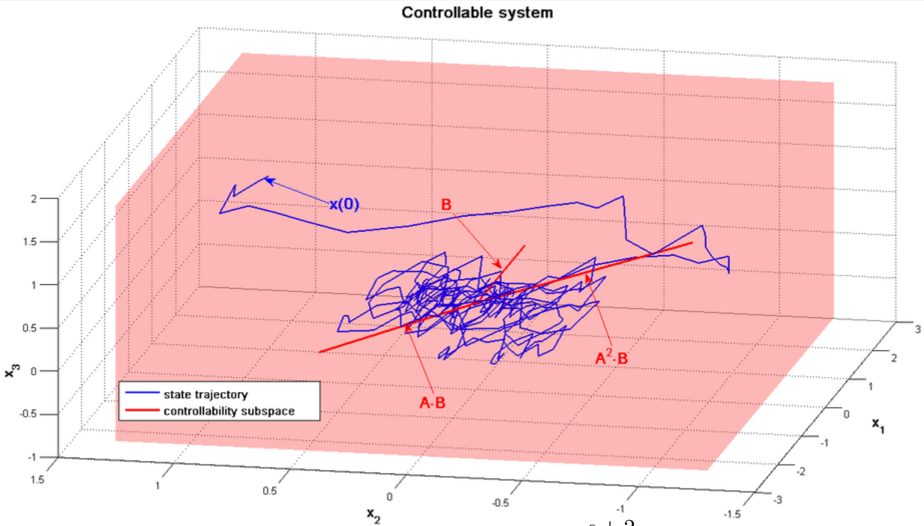
- $\boldsymbol{x}(0_+)$ is an arbitrary desired final state vector then we **can find a unique** $[g_1 ... g_n]^T$ **(for** $u(t)$**) iff** $rank\ \mathcal{C}_{n-1}(\boldsymbol{A}, \boldsymbol{B}) = n$.
- Controllability subspace: subspace spanned by the columns of $\mathcal{C}$
- Controllability is realization dependent since $\mathcal{C} = \mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$

Uncontrollable state space



System dynamics: $H(s) = \dfrac{s+2}{s^3+4s^2+6s+4}$

Controllable state space



System dynamics: $H(s) = \dfrac{s+2}{s^3 + 4s^2 + 6s + 4}$

Overview

DT controllability and reachability

Definition (controllability)

A discrete time system is said to be controllable if there exists a control sequence for each state such that the origin of the state space, that is $\boldsymbol{x}^* = 0$ (!!) can be reached in finite time.

Definition (reachability)

A discrete time system is said to be reachable (which is stronger that controllability) if it is possible to find a control sequence such that an arbitrary state $\boldsymbol{x}^*$ can be reached from any initial state $\boldsymbol{x}_0$ in finite time.

LTI controllability and reachability

- Controllability does not imply reachability
- Consider the solution of a DT-LTI state equation

$$\boldsymbol{x}(n) = \Phi^n \boldsymbol{x}(0) + \Phi^{n-1}\Gamma\boldsymbol{u}(0) + \ldots + \Gamma\boldsymbol{u}(n-1)$$

  with $\Phi^n \boldsymbol{x}(0) = 0$.

- *They are, however, equivalent if Φ is invertible*, i.e. it is of full rank.

Reachability of DT-LTI systems

Theorem (Reachability)

Given (Φ, Γ, C) *for*

$$\begin{aligned} \boldsymbol{x}(k+1) &= \Phi \boldsymbol{x}(k) + \Gamma \boldsymbol{u}(k), \quad \boldsymbol{x}(0) = \boldsymbol{x_0} \\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) \end{aligned}$$

This SSR is reachable if and only if the discrete controllability matrix $\boldsymbol{W_c}$

$$\boldsymbol{W_c} = [\ \Gamma \quad \Phi\Gamma \quad \dots \quad \Phi^{n-1}\Gamma\ ]$$

has full rank (n)*.*

A necessary and sufficient condition.

Reachability of DT-LTI systems

Proof.

(constructive)

- Given an initial condition $x(0)$. The solution of the state equation is

$$\boldsymbol{x}(n) = \Phi^n \boldsymbol{x}(0) + \Phi^{n-1}\Gamma \boldsymbol{u}(0) + \dots + \Gamma \boldsymbol{u}(n-1)$$
$$\boldsymbol{x}(n) = \Phi^n \boldsymbol{x}(0) + \boldsymbol{W_c}\mathcal{U}$$

  where

$$\boldsymbol{W_c} = [\ \Gamma \quad \Phi\Gamma \quad \dots \quad \Phi^{n-1}\Gamma\ ] \quad , \quad \mathcal{U} = [\ \boldsymbol{u}^T(n-1) \quad \dots \quad \boldsymbol{u}^T(0)\ ]^T$$

  $\boldsymbol{W_c}$ is the *discrete time controllability matrix*.

- We can design a suitable $\mathcal{U} \iff \boldsymbol{W_c}$ is of full rank.

$\square$

Discrete-time controllability

Example (Controllability, reachability)

A CT LTI SISO system is given by the following input-output model

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -1 & 1 \end{bmatrix} x(k)$$

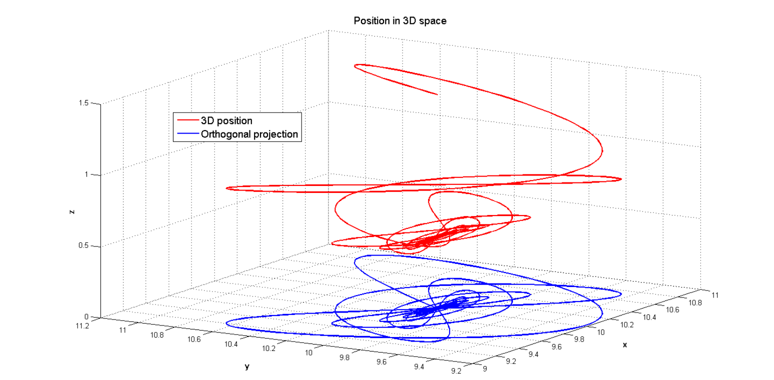
$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}(2) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

- Is the above state space model controllable, and or reachable?
- What is the relationship between the above two properties?
- Find an input signal, that governs the system from $\boldsymbol{x}(0)$ to $\boldsymbol{x}(2)$!

Overview

Observability of CT-LTI systems

- Problem statement
  - *Given*:
    - a state-space model with parameters $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$
    - a measurement record of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ as over a finite time interval
  - *Compute*:
    - The state signal $\boldsymbol{x}(t)$ over the finite time interval
    - It is enough to compute $\boldsymbol{x}(t_0) = \boldsymbol{x_0}$



Observability of CT-LTI systems

Theorem (Observability)

Given $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$. *This SSR with state space* $\mathcal{X}$ *is* state observable ***iff*** *the* observability matrix $\mathcal{O}_n$ *is of full rank*

$$\mathcal{O}_n = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}$$

Kalman rank condition: If $dim \mathcal{X} = n$ *then* $rank\ \mathcal{O}_n = n$.

- A necessary and sufficient condition

Observability of CT-LTI systems

Proof.

- Output and its derivatives can be expressed as

$$
\begin{aligned}
y &= \boldsymbol{C}\boldsymbol{x}\\
\dot{y} &= \boldsymbol{C}\dot{\boldsymbol{x}} = \boldsymbol{C}\boldsymbol{A}x + \boldsymbol{C}\boldsymbol{B}u\\
\ddot{y} &= \boldsymbol{C}\ddot{\boldsymbol{x}} = \boldsymbol{C}\boldsymbol{A}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u) + \boldsymbol{C}\boldsymbol{B}\dot{u} = \boldsymbol{C}\boldsymbol{A}^2x + \boldsymbol{C}\boldsymbol{A}\boldsymbol{B}u + \boldsymbol{C}\boldsymbol{B}\dot{u}\\
&.\\
&.\\
y^{(n-1)} &= \boldsymbol{C}\boldsymbol{x}^{(n-1)} = \boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{x} + \boldsymbol{C}\boldsymbol{A}^{n-2}\boldsymbol{B}u + \ldots + \boldsymbol{C}\boldsymbol{A}\boldsymbol{B}u^{(n-3)} + \boldsymbol{C}\boldsymbol{B}u^{(n-2)}
\end{aligned}
$$

- Matrix form

$$
\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ . \\ . \\ . \\ y^{(n-1)} \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C}\boldsymbol{A}^2 \\ . \\ . \\ . \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}
\boldsymbol{x} +
\begin{bmatrix}
0 & 0 & . & . & . & 0 \\
\boldsymbol{C}\boldsymbol{B} & 0 & . & . & . & 0 \\
\boldsymbol{C}\boldsymbol{A}\boldsymbol{B} & \boldsymbol{C}\boldsymbol{B} & 0 & . & . & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
\boldsymbol{C}\boldsymbol{A}^{n-2}\boldsymbol{B} & \boldsymbol{C}\boldsymbol{A}^{n-3}\boldsymbol{B} & . & . & \boldsymbol{C}\boldsymbol{B} & 0
\end{bmatrix}
\begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \\ . \\ . \\ . \\ u^{(n-1)} \end{bmatrix}
$$

$\square$

Observability of CT-LTI systems

- Compact form

$$\dot{\mathcal{Y}}(t) = \mathcal{O}_n x(t) + \mathcal{T}\dot{\mathcal{U}}(t)$$

- Zero initial state conditions

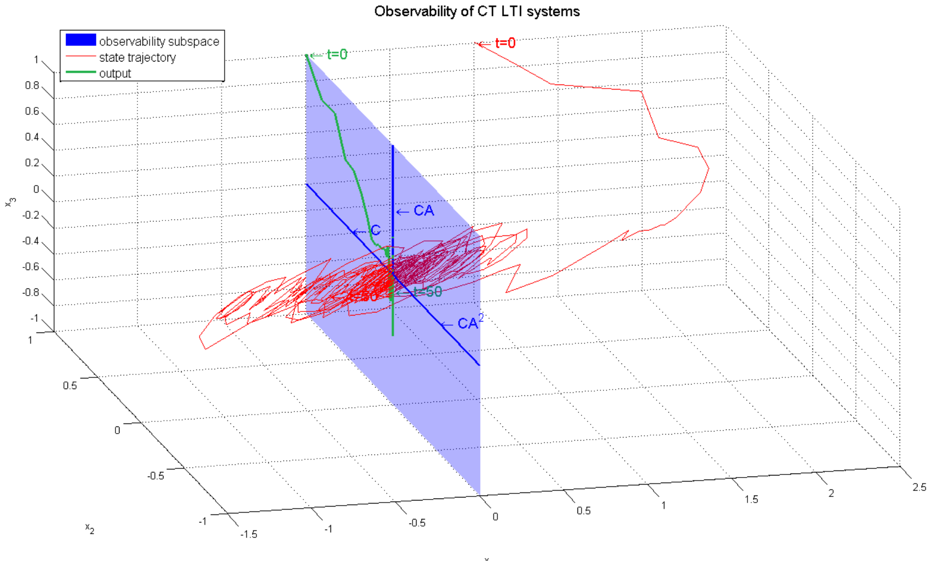
$$\dot{\mathcal{U}} = 0 \quad for \quad t = 0_-$$

- Then

$$\dot{\mathcal{Y}}(0_-) = \mathcal{O}_n(\boldsymbol{A}, \boldsymbol{C})\boldsymbol{x}(0_-)$$

- $x(0_-)$ can be uniquely determined iff $rank\ \mathcal{O}_n(\boldsymbol{A}, \boldsymbol{C}) = n$.
- Observability subspace: subspace spanned by the rows of $\mathcal{O}$
- Observability is realization dependent since $\mathcal{O} = \mathcal{O}(\boldsymbol{A}, \boldsymbol{C})$

Unobservable state space



Overview

Observability of DT-LTI systems

Definition (Observability)

A discrete time system is observable if there is a finite k such that the knowledge of

$$\{\boldsymbol{u}(0), ..., \boldsymbol{u}(k-1)\ ;\ \boldsymbol{y}(0), ..., \boldsymbol{y}(k-1)\}$$

with k being finite is sufficient to determine $\boldsymbol{x}(0)$.

Observability of DT-LTI systems

Theorem (Observability)

Given $(\Phi, \Gamma, \boldsymbol{C})$ *for*

$$\begin{aligned} \boldsymbol{x}(k+1) &= \Phi\boldsymbol{x}(k) + \Gamma\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) \end{aligned}$$

This system is observable if and only if the discrete observability matrix $\boldsymbol{W_o}$ *has full rank* (n)

$$\boldsymbol{W_o} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\Phi \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{C}\Phi^{n-1} \end{bmatrix}$$

A necessary and sufficient condition.

Observability of DT-LTI systems

Proof.

- Assume that $\boldsymbol{u}(k) = 0$ for $k = 0, 1, \ldots$. Then from $\boldsymbol{x}(k+1) = \Phi \boldsymbol{x}(k)$ we get

$$\begin{aligned}
\boldsymbol{y}(0) &= \boldsymbol{C}\boldsymbol{x}(0) \\
\boldsymbol{y}(1) &= \boldsymbol{C}\boldsymbol{x}(1) = \boldsymbol{C}\Phi\boldsymbol{x}(0) \\
&\vdots \\
\boldsymbol{y}(n-1) &= \boldsymbol{C}\Phi^{n-1}\boldsymbol{x}(0)
\end{aligned}$$

$$\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\Phi \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{C}\Phi^{n-1} \end{bmatrix} \boldsymbol{x}(0) = \begin{bmatrix} y(0) \\ y(1) \\ \cdot \\ \cdot \\ \cdot \\ y(n-1) \end{bmatrix}$$

with $\boldsymbol{W_o}$ being the *discrete time observability matrix*. □

Discrete-time observability

Example (Observability)

A CT LTI SISO system is given by the following input-output model

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \boldsymbol{x}(k)$$
$$y(t) = \begin{bmatrix} 1 & -2 \end{bmatrix} \boldsymbol{x}(t)$$

- Find the $\boldsymbol{x}(0)$ from the following output samples, if it is possible!

$$y(0) = 3, \quad y(1) = 2, \quad y(2) = 4$$