Discrete and Continuous Dynamical Systems

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Discrete and continuous dynamical systems:

Introduction to signals and systems

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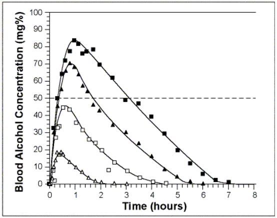
Overview

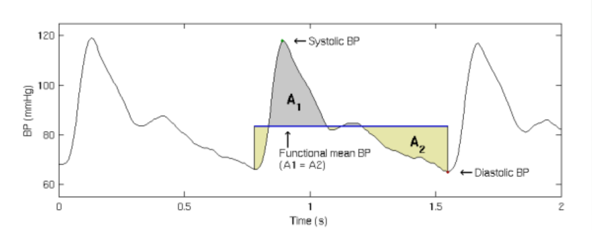
Signals – 1

Signal:
time-varying (and/or spatial varying) quantity

Examples

- $x : \mathbb{R}_0^+ \mapsto \mathbb{R}, \quad x(t) = e^{-t}$
- $y : \mathbb{N}_0^+ \mapsto \mathbb{R}, \quad y[n] = e^{-n}$
- $X : \mathbb{C} \mapsto \mathbb{C}, \quad X(s) = \frac{1}{s+1}$





Classification of signals

- dimension of the independent variable
- dimension of the signal
- real-valued vs. complex-valued
- continuous time vs. discrete time
- continuous valued vs. discrete valued
- bounded vs. unbounded
- periodic vs. aperiodic
- even vs. odd

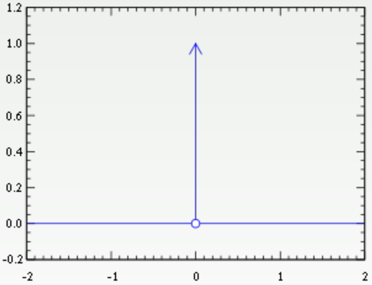
Special signals – 1

Dirac-δ or unit impulse function

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

where $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$ arbitrary smooth (many times continuously differentiable) function.
Consequence:

$$\int_{-\infty}^{\infty} 1 \cdot \delta(t)dt = 1$$

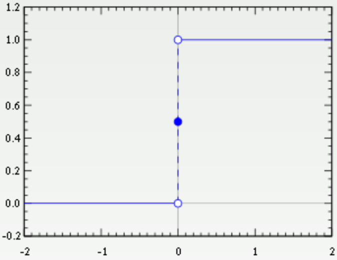


Physical meaning of the unit impulse:

- density impulse $\Rightarrow$ mass point

Special signals – 2

Unit step function



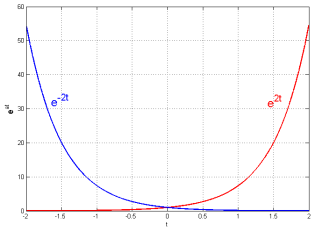
$$\eta(t) = \int_{-\infty}^{t} \delta(\tau) d\tau,$$

i.e.

$$\eta(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$$

Exponential function

$$e^{at}, \quad a \in \mathbb{R}$$



Complex exponential: $a \in \mathbb{C},\ a = \alpha + \mathrm{j}\Omega$

$$e^{at} = e^{\alpha t} \cdot e^{\mathrm{j}\Omega t} = e^{\alpha t}\cos(\Omega t) + \mathrm{j}e^{\alpha t}\sin(\Omega t)$$

Basic operations on signals – 1

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

- addition:
  $(\boldsymbol{x} + \boldsymbol{y})(t) = \boldsymbol{x}(t) + \boldsymbol{y}(t), \quad \forall t \in \mathbb{R}_0^+$
- multiplication by scalar:
  $(\alpha \boldsymbol{x})(t) = \alpha \boldsymbol{x}(t) \quad \forall t \in \mathbb{R}_0^+, \ \alpha \in \mathbb{R}$
- scalar product:
  $\langle \boldsymbol{x}, \boldsymbol{y} \rangle(t) = \langle \boldsymbol{x}(t), \boldsymbol{y}(t) \rangle \quad \forall t \in \mathbb{R}_0^+$

Basic operations on signals – 2

- time shift:
  $\mathbf{T}_a x(t) = x(t-a) \quad \forall t \in \mathbb{R}_0^+, a \in \mathbb{R}$
- convolution: $x, y : \mathbb{R}_0^+ \mapsto \mathbb{R}$

$$(x * y)(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau, \quad \forall t \geq 0$$

Laplace transform

Definition

The Laplace transform of a function $f(t)$ (i.e. $f : \mathbb{R}^+ \to \mathbb{R}$), is the function $F(s)$

$$F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$$

where $s = \sigma + \mathrm{i}\omega$. Alternative notation: $\mathcal{L}\{f\}$ instead of F

Definition

The inverse Laplace transform of a function of $F(s)$

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi\mathrm{i}} \lim_{T\to\infty} \int_{\gamma-\mathrm{i}T}^{\gamma+\mathrm{i}T} e^{st}F(s)\mathrm{d}s$$

Why is it good for us?

Laplace transform properties

Most important Laplace-transform properties:

Property/function	Time domain	s domain
Linearity	$af(t)+bg(t)$	$aF(s)+bF(s)$
Derivative	$f'(t)$	$sF(s)-f(0)$
Second derivative	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
Convolution	$\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$	$F(s)\cdot G(s)$
unit impulse	$\delta(t)$	1
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$
unit step	$\eta(t)$	$\frac{1}{s}$
exponential	$e^{\alpha t}$	$\frac{1}{s-\alpha}$

Overview

The concept of system

System

- An aggregation or assemblage of things so combined by nature or man as to form an integral or complex whole (*Encyclopedia Americana*).
- A regularly interacting or interdependent group of items forming a unified whole (*Webster's Dictionary*).
- A combination of components that act together to perform a function not possible with any of the individual parts (*IEEE Standard Dictionary of Electrical and Electronic Terms*).

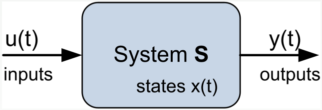
- consists of interacting components
- associated with a function to perform
- not always connected to objects and physical laws

System

System ($\mathbf{S}$): acts on signals

$$y = \mathbf{S}[u]$$

- inputs ($u \in \mathcal{U}$) and outputs ($y \in \mathcal{Y}$)
- abstract operator ($\mathbf{S} : \mathcal{U} \to \mathcal{Y}$)



Input-output modeling

- Measurable variables
  - Data: measuring it for $[t_0,\ t_f]$
  - Input variables can be manipulated

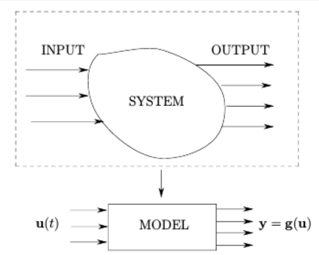
$$\{u_1(t), u_2(t), \ldots, u_p(t)\} \quad t_0 \geq t \geq t_f$$

  - Output variables can be directly measured

$$\{y_1(t), y_2(t), \ldots, y_m(t)\} \quad t_0 \geq t \geq t_f$$

  - Notation:

$$\boldsymbol{u}(t) = [u_1(t), u_2(t), \ldots, u_p(t)]^T$$
$$\boldsymbol{y}(t) = [y_1(t), y_2(t), \ldots, y_m(t)]^T$$

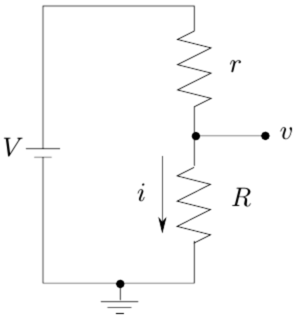


- Mathematical relationship

$$\left.\begin{array}{rcl} y_1(t) & = & g_1(u_1(t),\ \ldots, u_p(t)) \\ & \vdots & \\ y_m(t) & = & g_m(u_1(t),\ \ldots, u_p(t)) \end{array}\right\} \quad \boldsymbol{y} = \boldsymbol{g}(\boldsymbol{u})$$

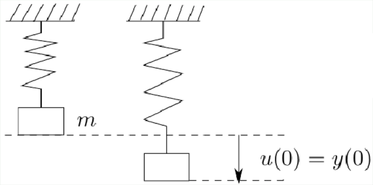
Examples

Example (Voltage divider)



$$v = V \frac{r}{r+R}$$
$$v = i\,R$$

Example (Spring-mass system)



$$m\,\ddot{y} = -k\,y$$
$$u(t) = \begin{cases} u_0 & t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Static and Dynamic Systems

Static system The output $y(t)$ is independent of the past inputs $(u(\tau),\ \tau < t)$

- algebraic equations
- memory not needed
- e.g. voltage divider

Dynamic system The output $y(t)$ dependens on the past inputs $(u(\tau),\ \tau < t)$ (difference equations)

- difference equations
- memory needed
- e.g. spring-mass system
- much more interesting

Time-Varying and Time-Invariant Systems

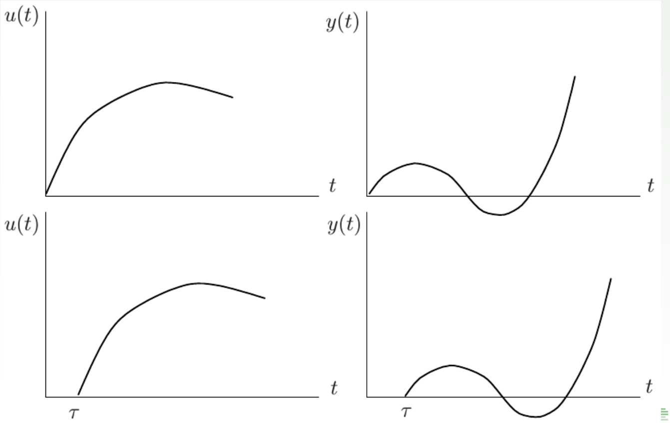
Is the output always the same when the same input is applied?

Time-varying $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{u}, t)$

- (some) parameters depend on times

Time-invariant $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{u})$

- constant parameters



The concept of state

Example (Spring-mass system)

Suppose, that

- $u(t)$ is known for $t \geq t_0$
- output $y(t)$ is observed at some $t = t_1 \geq t_0$

Is the above information enought to uniquely predict all future output $y(t), t > t_1$?

Definition (State)

The state of a system at time t_0 is the information required at t_0 s.t. the output $\boldsymbol{y}(t),\ \forall t > t_0$ is uniquely determined from this information and from $\boldsymbol{u}(t),\ t \geq t_0$.

State variable: $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T$

State Space

Definition (State equations)

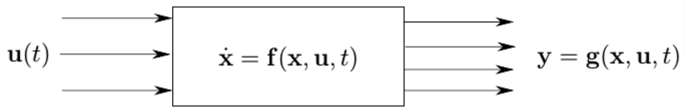
The set of equations required to specify the state $\boldsymbol{x}(t)$ for all $t \geq t_0$ given $\boldsymbol{x}(t_0)$and the function $\boldsymbol{u}(t)$, $t \geq t_0$, are called state equations.

Definition (State space)

The state space of a system, denoted by $\mathcal{X}$, is the set of all possible values that the state may take.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x_0} \qquad \text{(state equation)}$$

$$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \qquad \text{(output equation)}$$



Linear and Nonlinear Systems

Definition (Linear mapping)

The function $\boldsymbol{g}$ is said to be linear if and only if

$$\boldsymbol{g}(\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2) = \alpha_1 \boldsymbol{g}(\boldsymbol{u}_1) + \alpha_2 \boldsymbol{g}(\boldsymbol{u}_2)$$

Linear state space model

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \end{aligned}$$

Linear time-invariant state space model

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

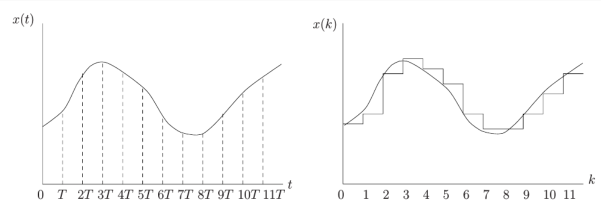
Continuous-State and Discrete-State Systems

Continuous	The state space $\mathcal{X}$ is a continuum
Discrete	The state space $\mathcal{X}$ is a discrete set
Hybrid	Some variables are dicrete, some are continuous

Discrete-Time Systems

Why?

- Digital computers operate in a discrete-time fashion, it has an internal discrete-time clock.
- Many differential equations of continuous-time models can only be solved numerically using a computer.
- Some systems are inherently discrete-time, e.g. economic models based on quarterly recorded data, etc.



Important: Discretization of time does not imply the discretization of the state space!

Discrete-time state space models

- Nonlinear

$$x(k+1) = f(x(k), u(k), k), \qquad x(0) = x_0$$
$$y(k) = g(x(k), u(k), k)$$

- Linear

$$x(k+1) = A(k)x(k) + B(k)u(k), \qquad x(0) = x_0$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

- Linear time-invariant

$$x(k+1) = Ax(k) + Bu(k), \qquad x(0) = x_0$$
$$y(k) = Cx(k) + Du(k)$$

Basic System properties

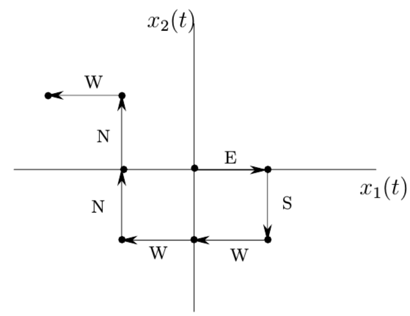
- SISO/MIMO
  Single Input-Single Output, or Multiple Input-Multiple Output system
- Continuous time (CT) vd discrete-time (DT) systems
  Continuous-time system: time set $\mathcal{T} \subseteq \mathbb{R}$
  Discrete-time system: time set $\mathcal{T} = \{\dots, t_{-1}, t_0, t_1, t_2, \dots\}$
- Causality
  The present depends only on the past, not on the future.

Overview

The Concept of Event

- Occurs instantaneously
- Causes state transition

Example (Random walk)



- A random step is taken in one of the four directions
- State is the position on the plane
- State space

$$X = \{(i,j) : i,j = \dots, -1, 0, 1, \dots\}$$

- Event set

$$E = \{N, S, W, E\}$$

Time-driven and event-driven systems

Time-driven At every clock tick an event e is to be selected from E. The state transitions are synchronized by the clock. The clock alone is responsible for any state transition.

Event-driven At various time instants some event e occurs. Every event $e \in E$ defines a distinct process through which the time instants when e occors are determined. State transitions are the results of combining of these asynchronous and concurrent processes.