Fuzzy control systems

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Introduction

- The notion of fuzziness:
- type of car – the determination is unambiguous
- speed of car – can be measured, but
  - the judgment is not unambiguous:
  - 50 km/h is high speed in a narrow street
  - 80 km/h is low speed in a highway
- the temperature:
  - 10 °C is high for an Eskimo
  - 10 °C is low for an African

Introduction

- Fuzzy controllers
  - classical control: the output value of the controller is based on the difference between the reference input and the measured output
  - fuzzy controllers: the determination of the output value of the controller is based on rules:

    if the speed is high **and** it begins to rain **then** reduce the speed

  - high, begin, reduce?? – has to be determined!

Fuzzy sets

- classical set theory – *crisp sets*:
  - elements of a set are members
  - the universe can be defined
  - the membership of an element of the universe in the set can be decided unambiguously
  - the size of a set is not restricted (empty set, infinite set)

- multisets: the same element can be member of set several times

Fuzzy sets

- fuzzy sets:
  - to assign a grade of membership to each element of the universe
  - membership grade: 0 – 1
  - obviously members: grade = 1
  - definitely not belong to the set: grade = 0
  - other members: 0 < grade < 1
  - there is no rule to determine the actual value of the grade

Fuzzy sets

- determination of membership
  - user knowledge relating to nature of the universe
  - often subjective
  - the membership criteria is described with linguistic variables (`high, medium, low`, e.g.)
  - the concept of the universe is similar to crisp sets
  - but the border between the set and its environment is not given clearly
  - elements with nonzero grade form the support of the fuzzy set

Fuzzy sets

- the membership function can be continuous or discrete
- most important continuous membership functions:
  - bell-shaped curves
  - s -curves
  - z -curves
  - π -curves
  - linear representations (straight lines or triangular shape)
  - irregularly shaped and arbitrary curves
  - discrete representation

Fuzzy sets

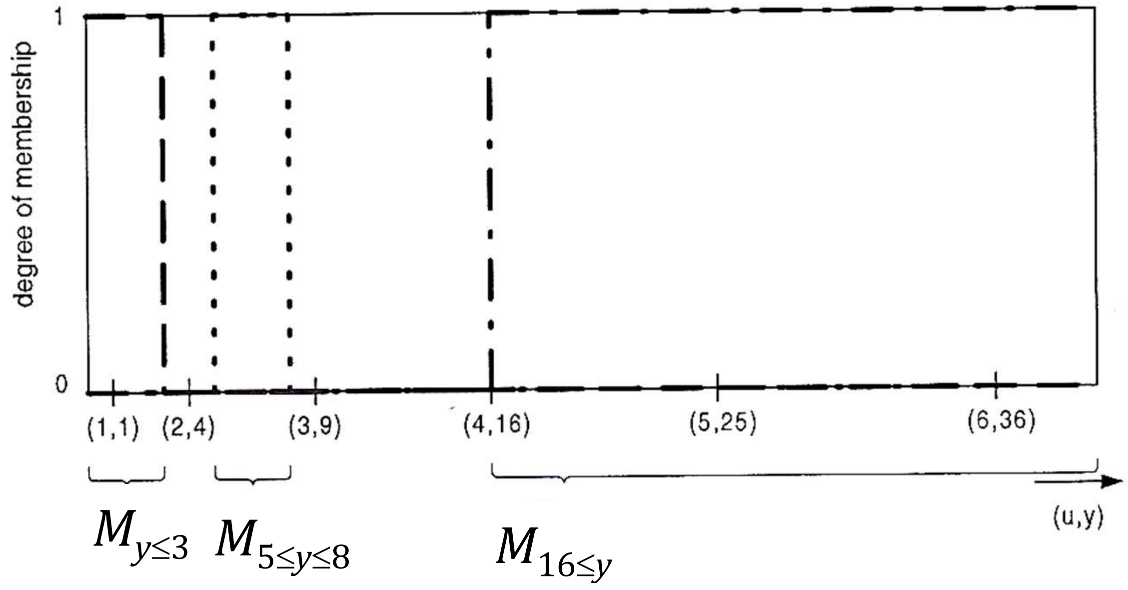
- membership function of crisp sets

$y = x^2$

u	1	2	3	4	5	…
y	1	4	9	16	25	…

$M_{y\leq 3} = \{\langle u,y\rangle \leq 3\}$ $\quad M_{y\geq 16} = \{\langle u,y\rangle \geq 16\}$ $\quad M_{5\leq y\leq 8} = \{5 \leq \langle u,y\rangle \leq 8\}$

$M_{y\leq 3} = \{\langle 1,1\rangle\}$ $\quad M_{y\geq 16} = \{\langle 4,16\rangle, \langle 5,25\rangle, \dots\}$ $\quad M_{5\leq y\leq 8} = \emptyset$



Fuzzy sets

- membership function of fuzzy sets
  - pairs considered to be high $M_{y>9}$

u	1	2	3	4	5	6
y	1	4	9	16	25	36
μ	0	0	0,2	0,6	1	1

  - pairs considered to be medium $M_{1<y<25}$

u	1	2	3	4	5	6
y	1	4	9	16	25	36
μ	0,1	0,4	0,9	0,8	0,1	0

Fuzzy sets

- pairs considered to be very low $M_{y<3}$

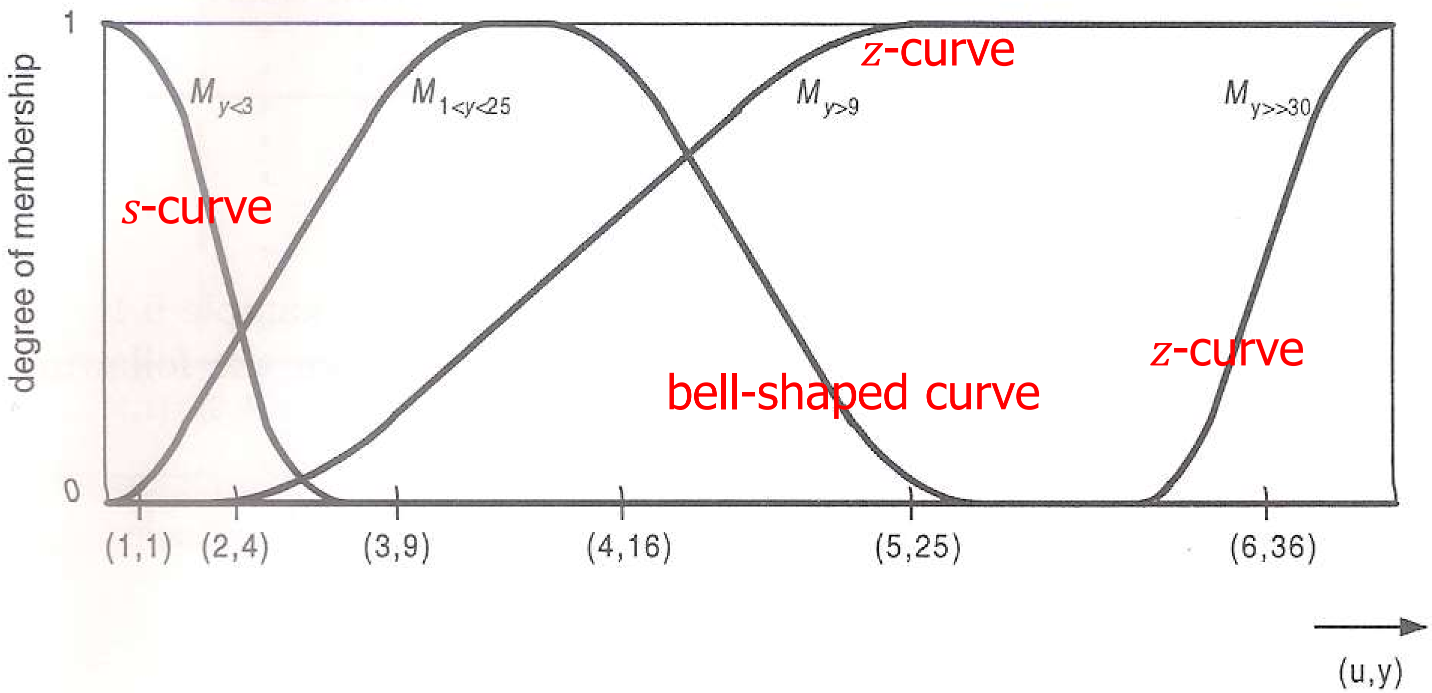
u	1	2	3	4	5	6
y	1	4	9	16	25	36
μ	1	0,4	0	0	0	0

- pairs where y is considered to be much higher 30

u	1	2	3	4	5	6
y	1	4	9	16	25	36
μ	0	0	0	0	0	0,5

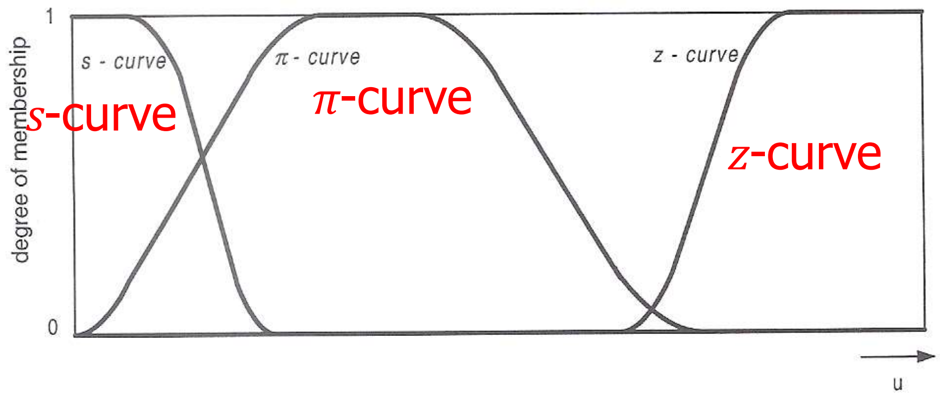
Fuzzy sets

- graphical representation of membership functions

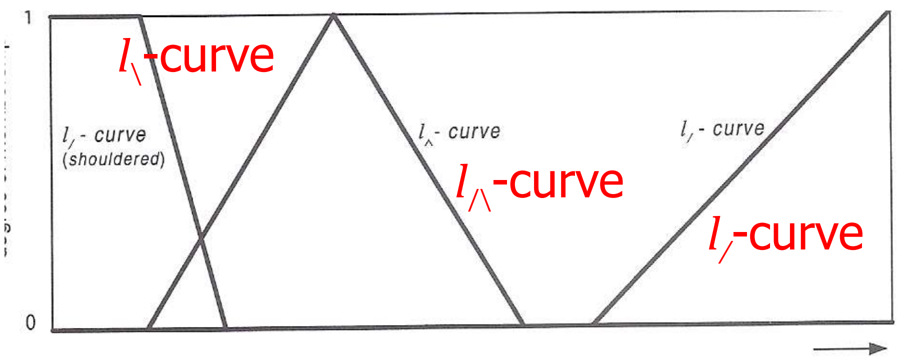


Fuzzy sets

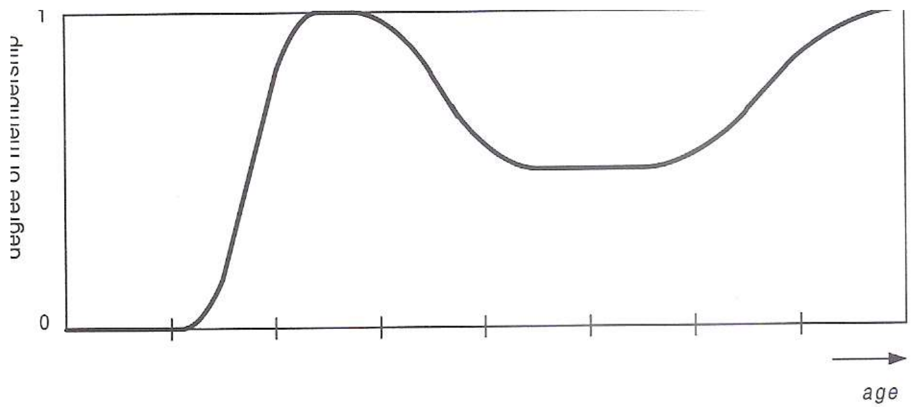
- s-, π- and z-curves



- linear representation



irregular curve



Fuzzy sets

- definition of fuzzy sets:

  fuzzy set A is a set of ordered pairs over the universe U

  $$A = \{\langle x, \mu(x) \rangle\}$$

  where $x \in U$ and $\mu(x)$ is its grade of membership in A.

- $\langle x, \mu(x) \rangle$ - fuzzy singleton
- representation of a fuzzy set as a vector:

  $$a = \mu_a = [\mu(x_1)\ \mu(x_2)\ \ldots\ \mu(x_n)]^T$$

Operations on fuzzy sets

- operations on crisp sets

$$A = \{1,2,3\} \qquad B = \{1,4,9\} \qquad U = \{0,1,2,3,4,5,6,7,8,9\}$$

  - the union of two sets

$$A \cup B = \{1,2,3,4,9\}$$

  - the intersection of two sets

$$A \cap B = \{1\}$$

  - the complement of set A

$$\neg A = \{0,4,5,6,7,8,9\}$$

Operations on fuzzy sets

- operations on fuzzy sets

  $A = \{\langle x, \mu(x)\rangle\}$ and $B = \{\langle y, \mu(y)\rangle\}$ fuzzy sets over the same universe U

  - the union of two sets

$$A \cup B = A \textbf{ or } B \equiv A \textbf{ max } B$$

$$\mu_{A\cup B}(x) = \textbf{max}\big(\mu_A(x), \mu_B(x)\big) \text{ for } \forall x \in U$$

  - the intersection of two sets

$$A \cap B = A \textbf{ and } B \equiv A \textbf{ min } B$$

$$\mu_{A\cup B}(x) = \textbf{min}\big(\mu_A(x), \mu_B(x)\big) \text{ for } \forall x \in U$$

Operations on fuzzy sets

- the complement of set A

$$\neg A = \textbf{not}\, A \equiv 1 - A$$

$$\mu_{\neg A}(x) = 1 - \mu_A(x) \quad \text{for } \forall x \in U$$

Operations on fuzzy sets

- truth tables

$$\mu_A, \mu_B \in \{0;\ 0{,}25;\ 0{,}5;\ 0{,}75; 1{,}00\}$$

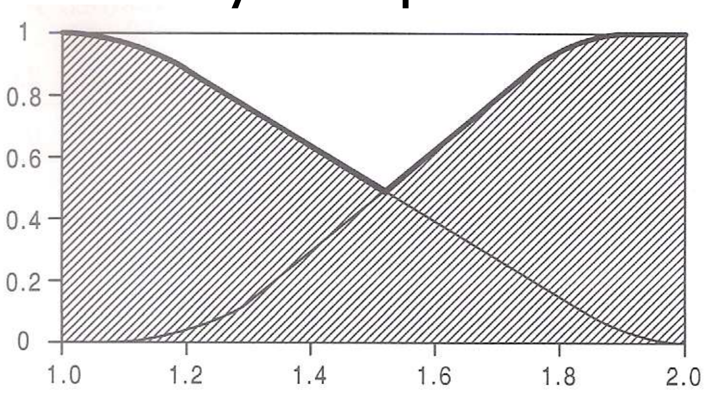
or	0,00	0,25	0,50	0,75	1,00
0,00	0,00	0,25	0,50	0,75	1,00
0,25	0,25	0,25	0,50	0,75	1,00
0,50	0,50	0,50	0,50	0,75	1,00
0,75	0,75	0,75	0,75	0,75	1,00
1,00	1,00	1,00	1,00	1,00	1,00

and	0,00	0,25	0,50	0,75	1,00
0,00	0,00	0,00	0,00	0,00	0,00
0,25	0,00	0,25	0,25	0,25	0,25
0,50	0,00	0,25	0,50	0,50	0,50
0,75	0,00	0,25	0,50	0,75	0,75
1,00	0,00	0,25	0,50	0,75	1,00

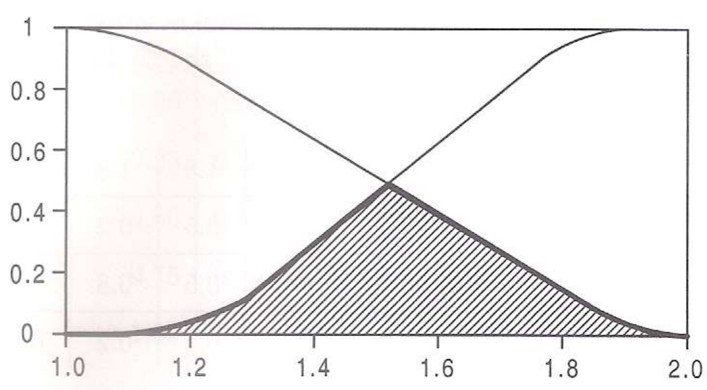
Operations on fuzzy sets

- graphical representation of fuzzy operators

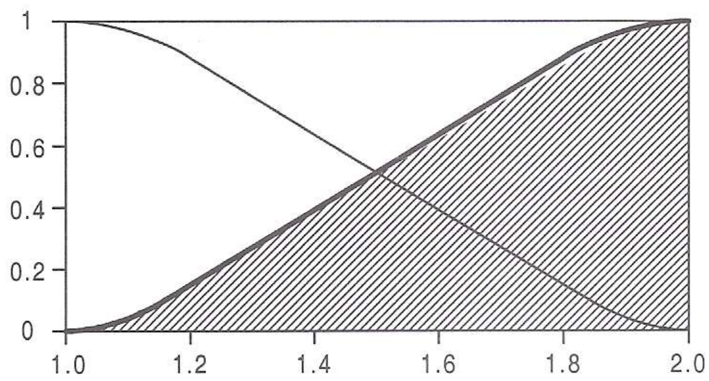
fuzzy **or** operator



fuzzy **and** operator



fuzzy **not** operator



Operations on fuzzy sets

- properties of fuzzy operators:
  - commutativity $a \textbf{ or } b = b \textbf{ or } a$

    $a \textbf{ and } b = b \textbf{ and } a$
  - associativity $(a \textbf{ or } b) \textbf{ or } c = a \textbf{ or } (b \textbf{ or } c)$

    $(a \textbf{ and } b) \textbf{ and } c = a \textbf{ and } (b \textbf{ and } c)$
  - distributivity $a \textbf{ or } (b \textbf{ and } c) = (a \textbf{ or } b) \textbf{ and } (b \textbf{ or } c)$

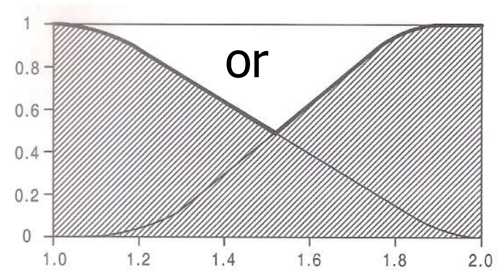
    $a \textbf{ and } (b \textbf{ or } c) = (a \textbf{ and } b) \textbf{or } (b \textbf{ and } c)$
  - DeMorgan $\textbf{not } (b \textbf{ and } c) = (\textbf{not } a) \textbf{ or } (\textbf{not } b)$

    $\textbf{not } (a \textbf{ or } b) = (\textbf{not } a) \textbf{ and } (\textbf{not } b)$

Operations on fuzzy sets

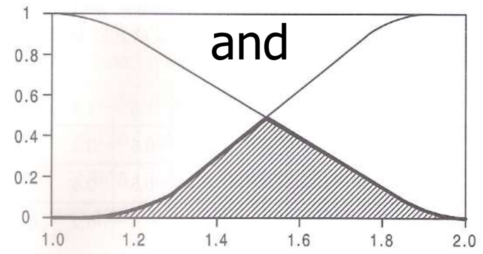
- absorption $(a \textbf{ and } b) \textbf{ or } a = a$

  $(a \textbf{ or } b) \textbf{ and } a = a$



- idempotency $a \textbf{ or } a = a$

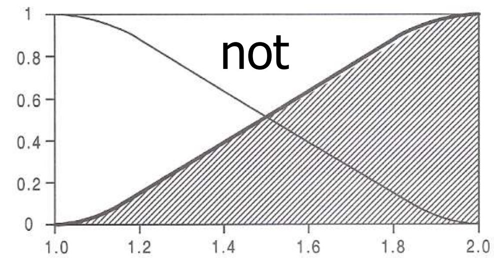
  $a \textbf{ and } a = a$



but

- exclusion $a \textbf{ or } \neg a \neq 1$

  $a \textbf{ and } \neg a \neq \emptyset$



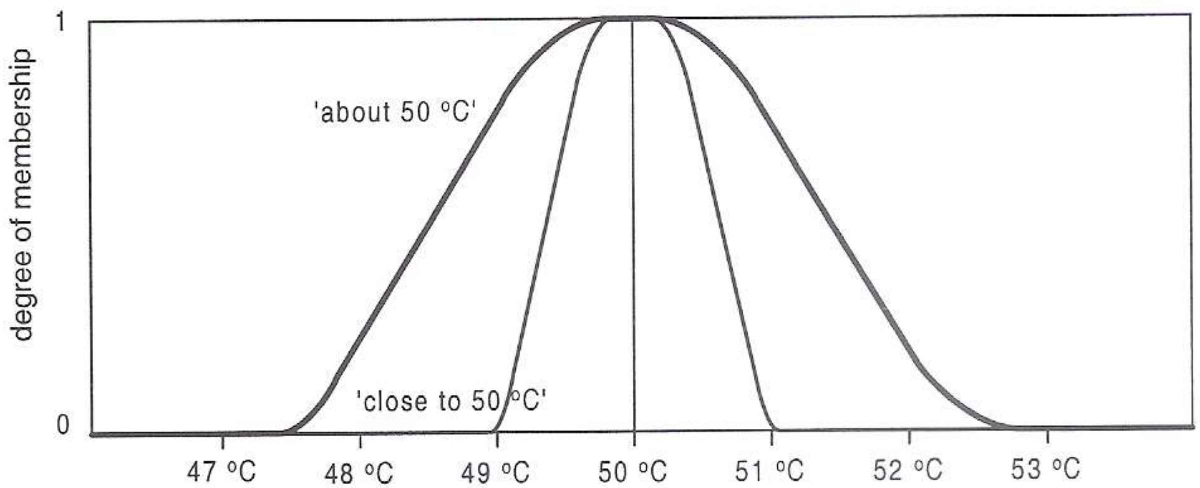
Operations on fuzzy sets

- linguistic modifiers
  - approximation of fuzzy sets
    - about, around, close to
  - restriction of fuzzy sets
    - below, above
  - intensification and dilution of fuzzy sets
    - very, extremely $\mathbf{int}\,\mu(x) = \mu^n(x) \quad n = 2, 3$
    - somewhat, greatly $\mathbf{dil}\,\mu(x) = \mu^{1/n}(x) \quad n = 2, 1.4$

Operations on fuzzy sets

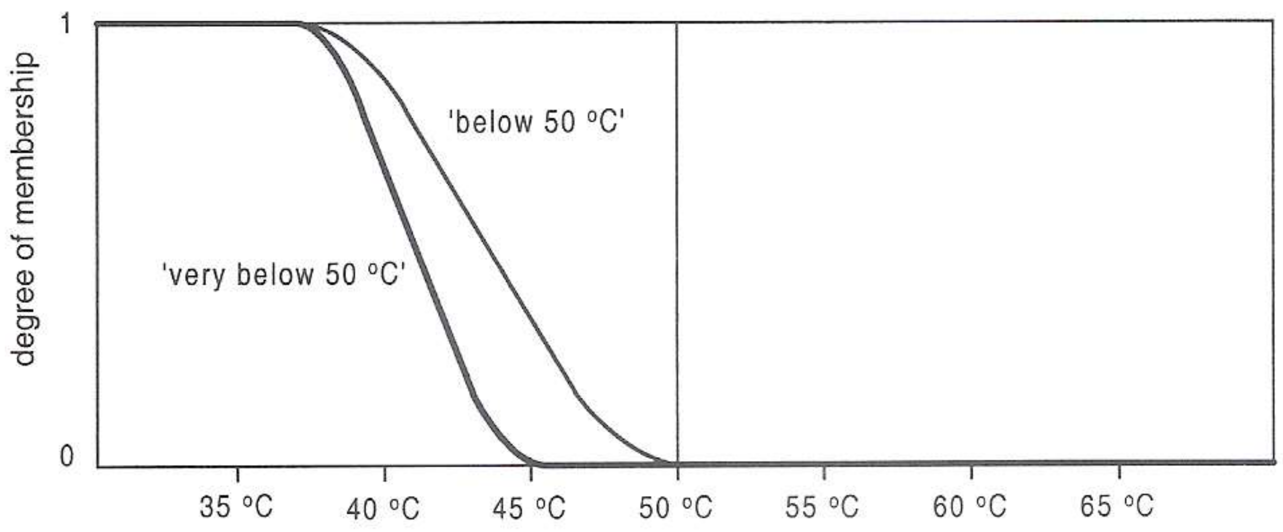
- graphical representation of linguistic operators

  - keep the controlled variable **about** 50°C
  - keep the controlled variable **close to** 50°C



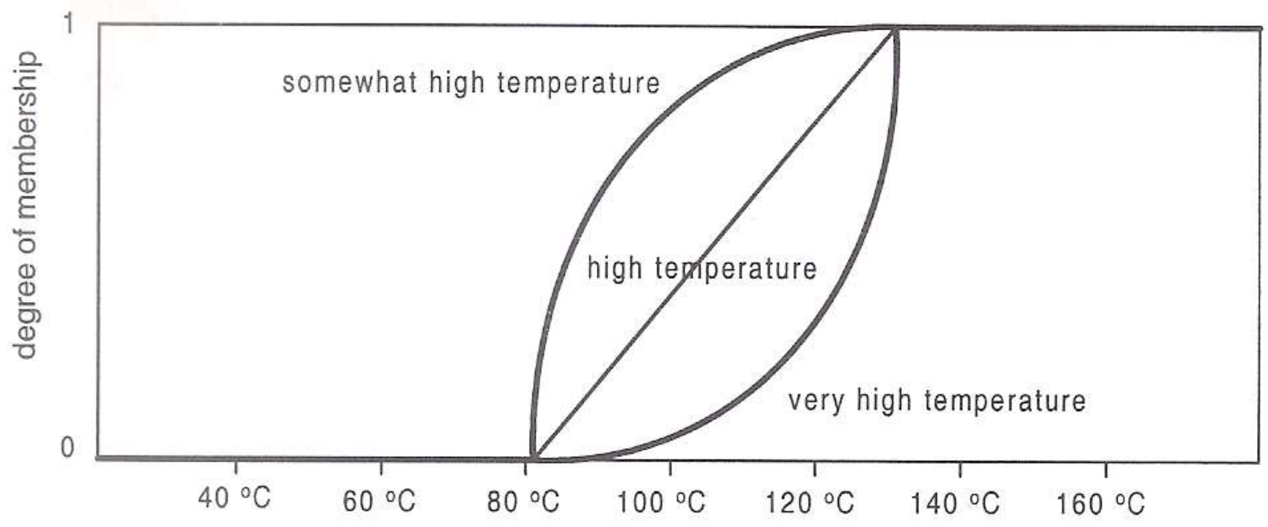
Operations on fuzzy sets

- keep the controlled variable **below** 50°C
- keep the controlled variable **very below** 50°C



Operations on fuzzy sets

- high temperature with linear representation
- very high temperature
- somewhat high temperature



Inference on fuzzy sets

- Relation between fuzzy sets
  - relation:

P, Q, S – universes

$p \in P$, $q \in Q$ events, $s_1, s_2 \in S$ states

R_1 fuzzy relation: an event a causes state b in a given degree

R_2 fuzzy relation: a state b is a precondition of event a in a given degree

Inference on fuzzy sets

- the relation between P and S:

$\mathbf{R_1}$	s_1	s_2
p	0.3	0.9

- the relation between Q and S:

$\mathbf{R_2}$	q
s_1	0.9
s_2	0.7

Inference on fuzzy sets

- conclusions

(event p causes the state s_1 in degree 0.3)

and

(state s_1 is a precondition of event q in degree 0.9)

(event p causes the state s_2 in degree 0.9)

and

(state s_2 is a precondition of event q in degree 0.7)

$$\Downarrow$$

(event p generates q in degree 0.3)

or

(event p generates q in degree 0.7)

Inference on fuzzy sets

- formal definition of fuzzy relations
  - composition of binary fuzzy relation

  Given two fuzzy sets in matrix form. Their compositions is:

$$W = U \circ V$$

$$w_{i,j} = (u_{i1} \wedge v_{1j}) \vee (u_{i2} \wedge v_{2j}) \vee \cdots \vee (u_{ip} \wedge v_{pj})$$
$$= \bigvee_{k=1}^{p} u_{ik} \wedge v_{kj}$$

where $\circ$ is an **inner or–and product** (or max-min composition).

Inference on fuzzy sets

- implication on fuzzy sets
  - $U_e = \{-5, -2.5, 0, 2.5, 5\}$ $U_u = \{-2, 0, 2\}$
  - fuzzy sets:

large positive error	$\mu_{lpe} = [0\ 0\ 0\ 0.6\ 1]^T$
small positive error	$\mu_{spe} = [0\ 0\ 0.3\ 1\ 0.3]^T$
zero error	$\mu_{ze} = [0\ 0.3\ 1\ 0.3\ 0]^T$
small negative error	$\mu_{sne} = [0.3\ 1\ 0.3\ 0\ 0]^T$
large negative error	$\mu_{lne} = [1\ 0.3\ 0\ 0\ 0]^T$
positive control signal	$\mu_{pcs} = [0\ 0.2\ 1]^T$
zero control signal	$\mu_{zcs} = [0.1\ 1\ 0.1]^T$
negative control signal	$\mu_{ncs} = [1\ 0.2\ 0]^T$

Inference on fuzzy sets

- control rule:

$$e \rightarrow u$$

  - let the error signal $e = 5$, (large positive error)
  - positive control signal

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.6 \\ 1 \end{bmatrix} \times [0 \quad 0.2 \quad 1] = \begin{bmatrix} 0 \wedge 0 & 0 \wedge 0.2 & 0 \wedge 1 \\ 0 \wedge 0 & 0 \wedge 0.2 & 0 \wedge 1 \\ 0 \wedge 0 & 0 \wedge 0.2 & 0 \wedge 1 \\ 0.6 \wedge 0 & 0.6 \wedge 0.2 & 0.6 \wedge 1 \\ 1.0 \wedge 0 & 1 \wedge 0.2 & 1 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0.6 \\ 0 & 0.2 & 1 \end{bmatrix}$$

Inference on fuzzy sets

- formal definition of implication on fuzzy sets
  - let A and B be two fuzzy set, then the implication:

$$A \to B \triangleq A \times B$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & \dots & b_m \end{bmatrix} =$$

$$= \begin{bmatrix} a_1 \wedge b_1 & a_1 \wedge b_2 & \dots & a_1 \wedge b_m \\ a_2 \wedge b_1 & a_2 \wedge b_2 & \dots & a_2 \wedge b_m \\ \vdots & \vdots & & \vdots \\ a_n \wedge b_1 & a_n \wedge b_2 & \dots & a_n \wedge b_m \end{bmatrix}$$

Inference on fuzzy sets

- inference on fuzzy sets
  - rule-base: set of rules in the form of implications

$$A \rightarrow B$$

  - if A is true, all the rules containing this statement in their conditional parts is needed to determine the necessary action
  - generalized modus ponens

$$\frac{A', A \rightarrow B}{B'}$$

where A' similar to A, B' similar to B

Inference on fuzzy sets

- compositional rule reference
  - Let R be relation between U_1 and U_2 and A a fuzzy set over U_1. Then the compositional rule:

    $$A \circ R = B$$

    where $\circ$ the composition operator and B a fuzzy set on U_2.
  - inner matrix product

Inference on fuzzy sets

- let R be defined between **lpe** and **pcs** :

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0.6 \\ 0 & 0.2 & 1 \end{bmatrix}$$

- apply **somewhat** on **lpe**:

$$\textbf{somewhat lpe} = [0 \quad 0 \quad 0 \quad 0.6 \quad 1]^{1/2}$$
$$= [0 \quad 0 \quad 0 \quad 0.77 \quad 1]$$

Inference on fuzzy sets

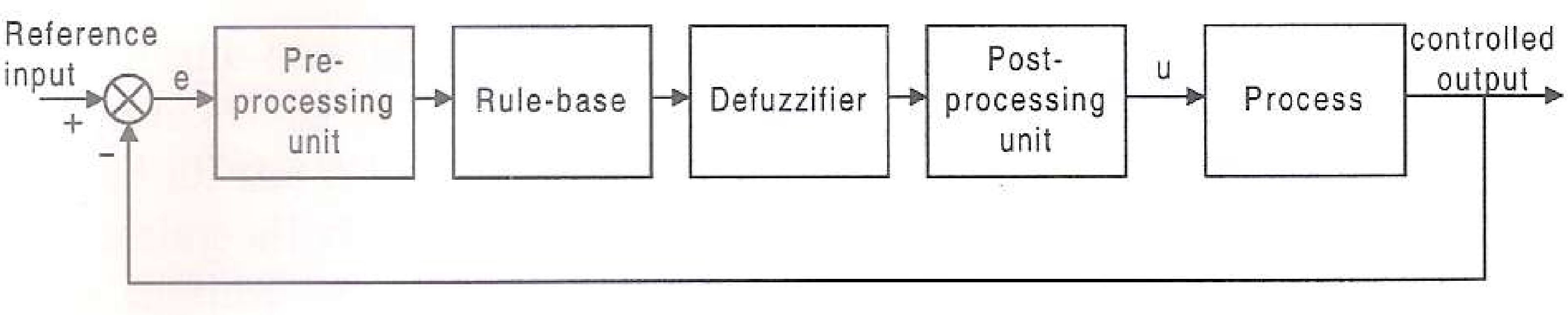
- e has somewhat large positive value (**lpe**')
- necessary interaction: **pcs**'
- determination of **pcs**' based on the relation R:

$$R: \mathbf{lpe} \rightarrow \mathbf{pcs}$$

$$\mathbf{pcs}' = [0 \quad 0 \quad 0 \quad 0.77 \quad 1] \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0.6 \\ 0 & 0.2 & 1 \end{bmatrix} = [0 \quad 0.2 \quad 1]$$

Fuzzy controllers

- general structure of a rule based fuzzy control system:



- multi input – multi output process control

Fuzzy controllers

- elements of controller:
  - preprocessing unit: convert the error signal (crisp data) into fuzzy form
  - rule-base: inferencing, determination of the necessary control action
  - defuzzifier: convert the determined control action back into crisp value
  - postprocessing unit: tuning and amplifying the signal

Fuzzy controllers

- design of fuzzy controller:
  - direct controller design: fuzzy controller is designed without modelling the process
  - design of process model: first the fuzzy model of controlled system is done, then it is used to model the controller

- fuzzy controller types: fuzzy PID controller, table-based controller, self-organizing controller, neuro-fuzzy controller

Fuzzy controllers

- input and output signals of the fuzzy controller
  - important, defines the structure of the controller
  - typical inputs: error signal, derivates and integrals of error
  - depend on the system properties (dynamics, stability, time dependency, nonlinearity,…)
  - MIMO systems: state variables, noises,…
  - too much signal $\rightarrow$ more complex rule-base
  - output value: absolute value or incremental value of the control signal

Fuzzy controllers

- determination of the universes
  - depends on the system to be controlled
  - minimum and maximum value – operating range
  - resolution – accuracy and calculation requirements
  - standardized ranges: [-1,1], [-100,100]
  - scaling factors, zero level

Fuzzy controllers

- determination of the membership functions
  - number and shape of membership functions
  - number?: rule base complexity $\leftrightarrow$ flexibility
    - rule of thumb: 2, 3, 5
      - 2: negative, positive
      - 3: small, medium, high
      - 5: very small, …, very high
  - shape?: continuous $\leftrightarrow$ discrete
    - continuous: several shapes, better description, more time for inferencing
    - discrete: vectors, accuracy

Fuzzy controllers

- scalar as input value
  - singleton
  - special fuzzy set: grade of membership function: 0;1
  - inferencing is simpler
  - rules more intuitive
- membership functions:
  - number: 3
  - triangular shape (symmetrical, similar, shouldered)
  - base of the triangles enough wide

Fuzzy controllers

- Rule-base
  - rules: **`if-then`** format
  - presentation to end-users:
    - linguistic description
    - relational, tabular format
    - graphic representation
  - possibilities to find the rules

Fuzzy controllers

- normalized (standard) rule base
  - for fuzzy P, PI, PD, PID controllers

PD		Δe				
		LN	SN	ZERO	SP	LP
e	LN	ln	ln	sn	sn	nc
	SN	ln	sn	sn	nc	sp
	ZERO	sn	sn	nc	sp	sp
	SP	sn	nc	sp	sp	lp
	LP	nc	sp	sp	lp	lp

If the error signal is equal to zero
and the its change is small negative
then the control signal is small negative

- easy and understandable in case of nonlinear system control

Fuzzy controllers

- experience and intuition of experts
  - rules from operator's handbook, logbook
  - interview the operators
- the fuzzy model of the process
  - model of the system $\rightarrow$ inverse of the model of the controller
- learning type controller
  - self-organizing, neuro-fuzzy controllers

Fuzzy controllers

- rule-base analysis
  - well designed rule-base is the requirement of the proper operation of fuzzy control
  - most important properties:
    - completeness
    - consistency
    - redundancy
    - interaction

Fuzzy controllers

- completeness
  - if every non-zero input generates a non-zero output
  - reasons of incompleteness
    - gap between membership functions
    - missing rules
  - checking
    - graphic representation of membership functions
    - let X_i be the conditional part of i-th rule:

$$\left(\bigvee X_i\right) > \varepsilon \qquad 0 < \varepsilon \leq 1$$

Fuzzy controllers

- consistency
  - a rule-base is inconsistent if two or more rules with same or very similar conditional parts generate different outputs
  - slightly different input parts generate slightly different output sets
  - measurement the differences between input and output parts:

$$m_{ij} = (X_i \text{ similar\_to } X_j) \text{ and not}(U_i \text{ similar\_to } U_j)$$

  - **similar_to**: degree of similarity based on the overlap between the two fuzzy sets

Fuzzy controllers

- redundancy
  - if there are at least two rules with the same or very similar **if-then** parts
  - reasons:
    - adding the same rule twice
    - new rule to be added but already covered by an existing rule
  - storage and computing problem, but not inconsistency

Fuzzy controllers

- rule is redundant if its sets are subsets of another rule:

$$\mu_i = R_i \textbf{ in } (R \backslash R_i)$$

where R_i is a rule in rule-base R and $R = \bigvee R_j$

- measure the redundancy:

$$R' = R \setminus R_i = \bigvee R_j\,, \qquad j = 1 \ldots n, \text{but } j \neq i$$

- if the elements of matrix R' are greater or equal to the elements of matrix R_i then rule R_i is redundant

Fuzzy controllers

- interaction
  - independency of the conditional parts of rules
  - if the input relations of these conditional parts are disjoint → no interaction between rules
  - overlap between the input relations → the inferred output set may not be equal to output part of a given rule
  - not a general requirement
  - degree of interaction

$$\nu_i = \|(X_i \circ R) - U_i\|$$

Fuzzy controllers

- The operation of fuzzy controller
  - preprocessing unit – fuzzification
    - convert the output signal of the system into input data for the inferencing process $\rightarrow$ grades of membership for the conditional parts of the rules
    - first the output signal of the system have to scaled to the standardized universes
    - then grades of membership have to be determined for all membership functions related to the given variable

Fuzzy controllers

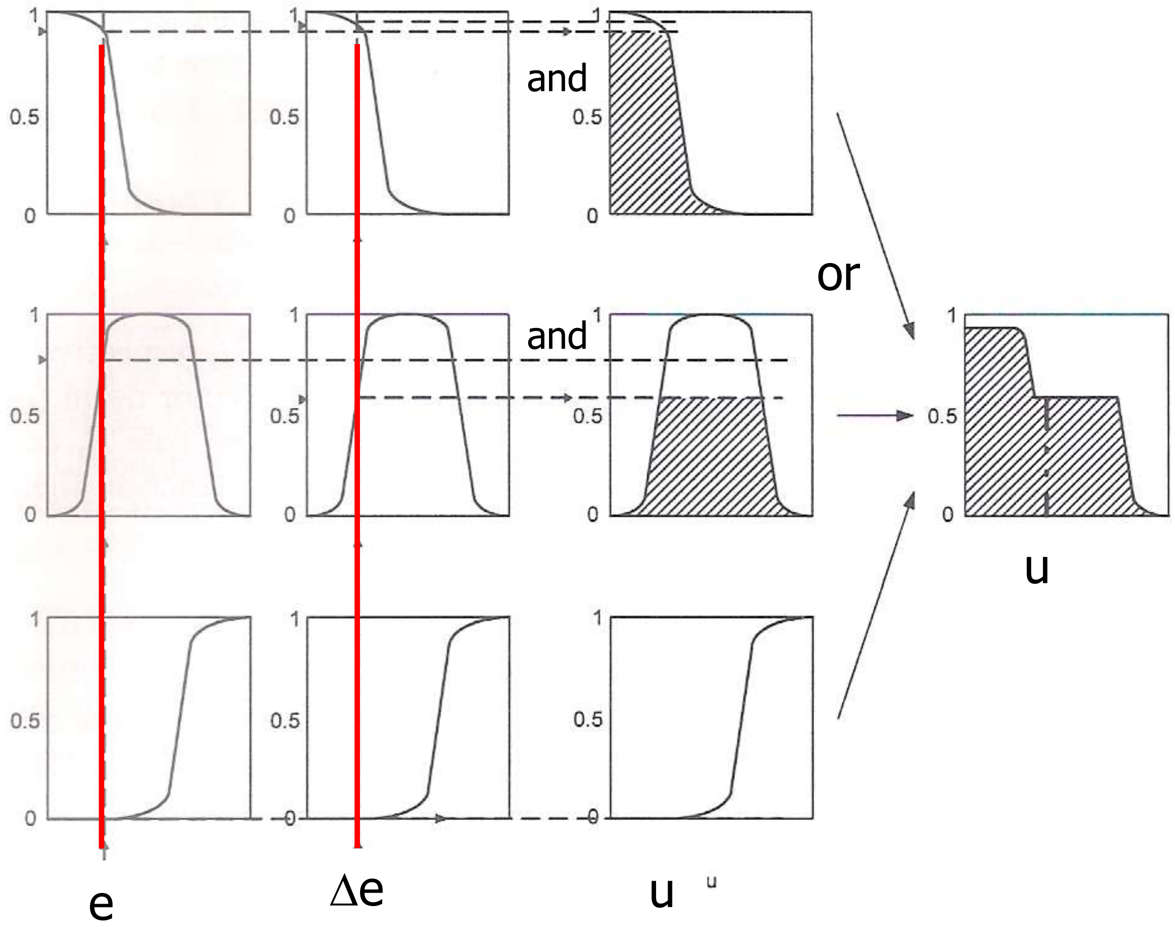
- the inference engine
  - inference: what extent each rule is fulfilled
  - assume the following rules:

if e is small negative and Δe is large negative then u is large negative

if e is zero and Δe is large negative then u is small negative

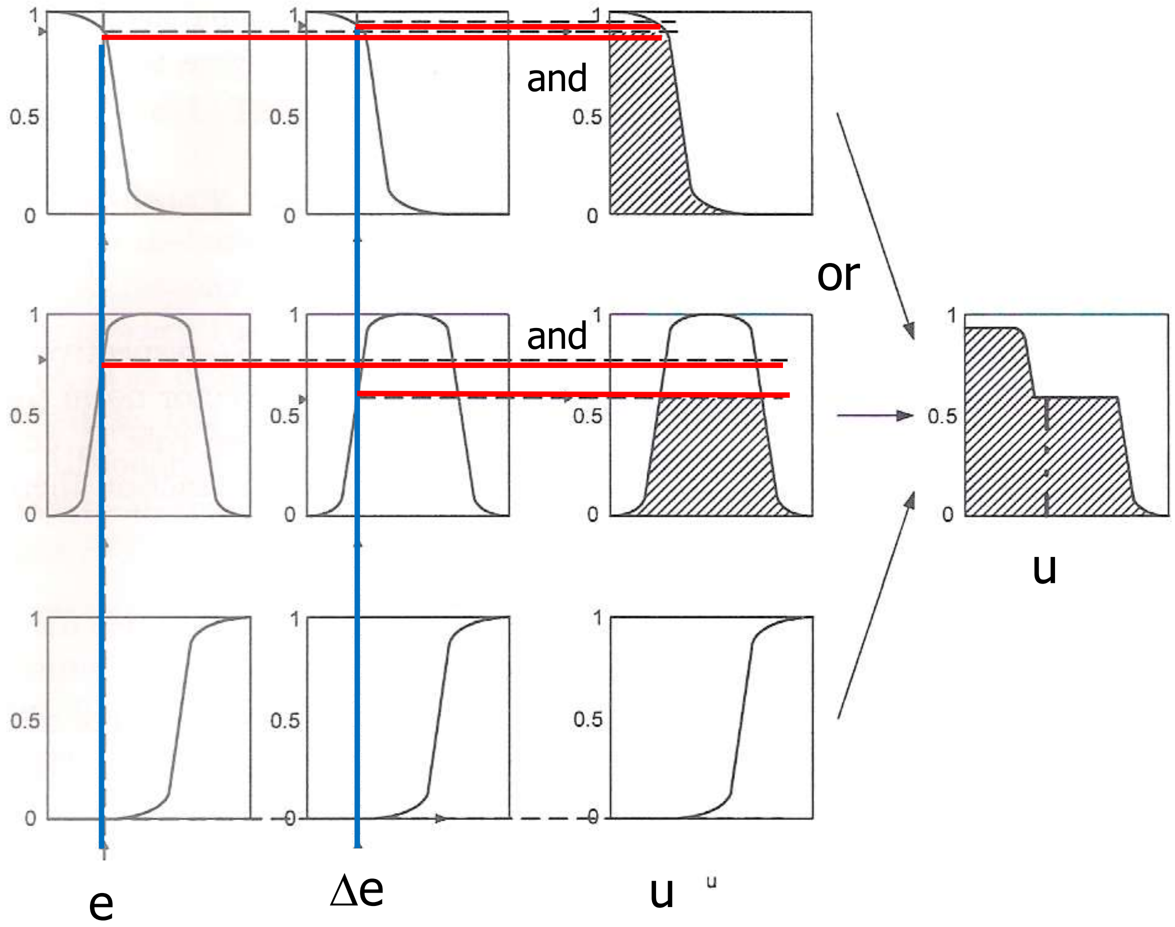
Fuzzy controllers

Step 1 preprocessing: determination of membership grade – vertical lines in the first two columns



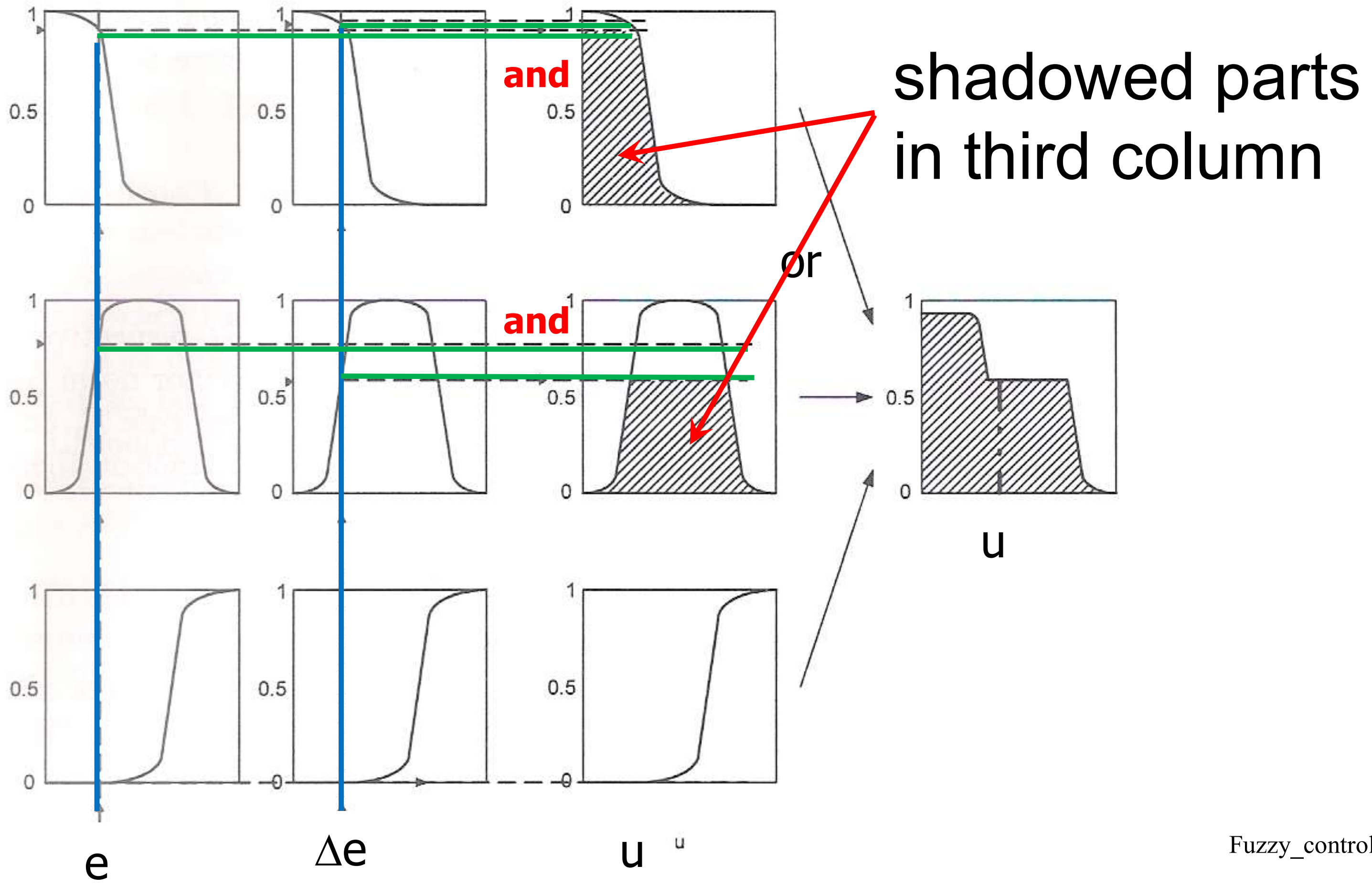
Fuzzy controllers

Step 2 inference engine: determination of the membership grade of each term in the conditional part of the rules – horizontal lines in the first two columns



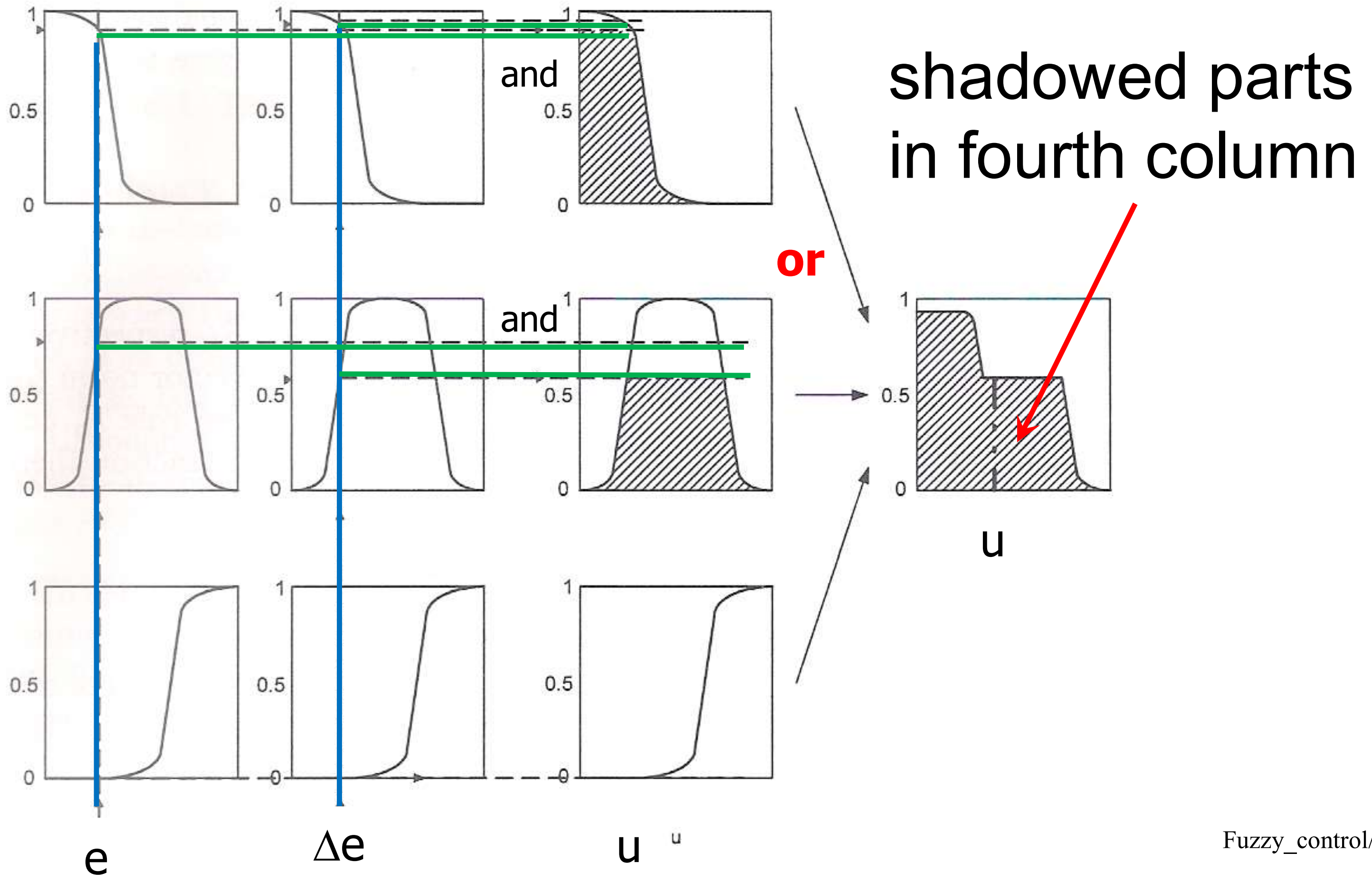
Fuzzy controllers

Step 3 using the operation **min** (fuzzy **and**) the inference engine determines the grade of fulfillment for conditional parts of each rule and implies the contribution of the rule to the output value



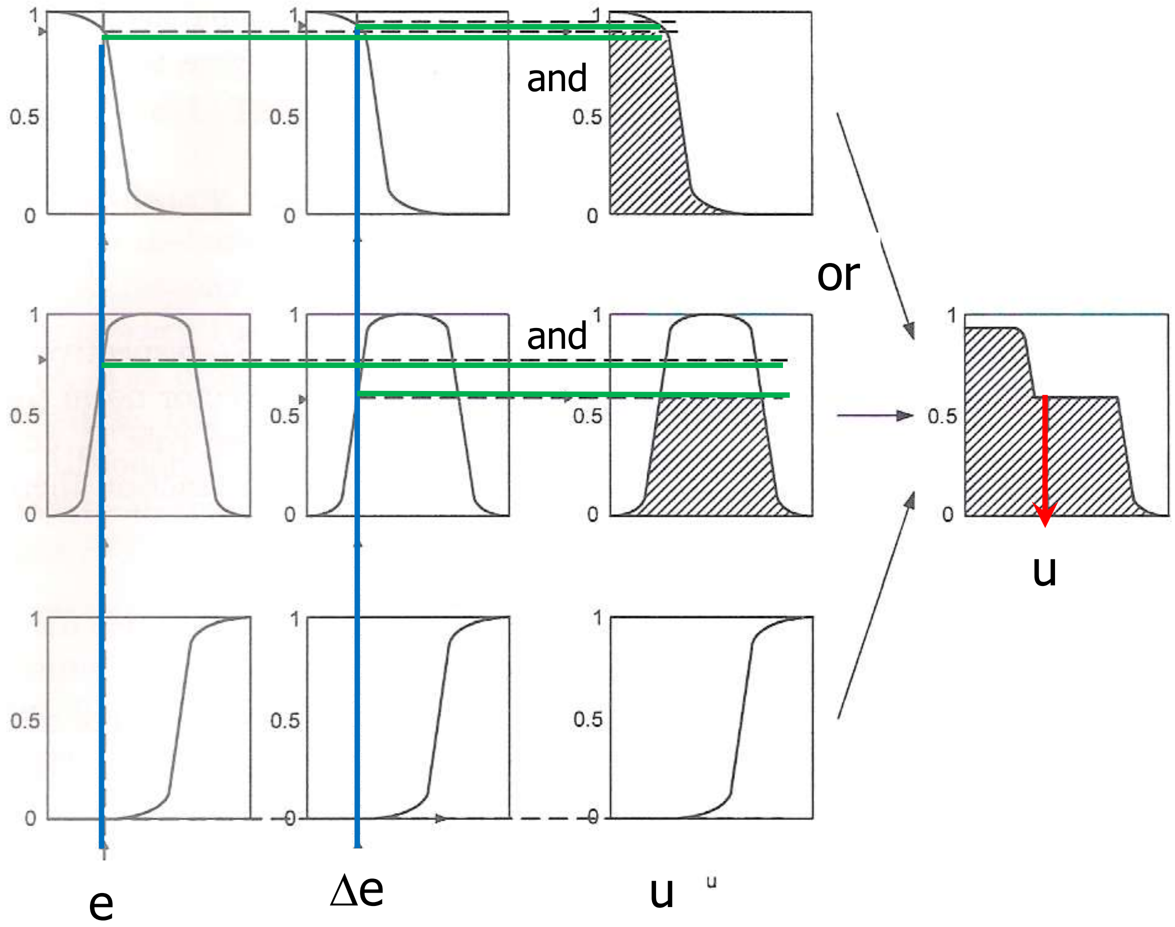
Fuzzy controllers

Step 4 collecting all contributions and using **max** (fuzzy **or**) the resulting fuzzy set is determined – fourth column



Fuzzy controllers

Step 5 postprocessing: the resulting fuzzy set has to be converted into crisp value – Centre of area method in fourth column



Fuzzy controllers

- postprocessing unit - defuzzification
  - convert the fuzzy set into a crisp control signal
  - *mean of maxima* – maximum possible value or the averages of maximum values

$$u = \frac{\sum_{j=1}^{l} x_{mj}}{l}$$

where

x_{mj} denotes the maximum value of the j -th term

l number of terms

Fuzzy controllers

- *centre of area method* – value which divides the fuzzy sets into two part with equal areas

  in case of discrete membership functions:

$$u = \frac{\sum_{j=1}^{l} \mu(x_j) x_j}{\sum_{j=1}^{l} \mu(x_j)}$$

where

$\mu(x_j)$ is the membership grade of the j -th term at the value x_j of the discrete universe

l number of terms

Fuzzy controllers

- *selecting the maximum value* – select the term with the maximum membership grade
leftmost maximum, rightmost maximum

- *height* – for singleton type outputs the step of inference and defuzzification can be combined:

$$u = \frac{\sum_{j=1}^{l} \alpha_j s_j}{\sum_{j=1}^{l} \alpha_j}$$

where

s_j is the value of the j -th singleton
and α_j is its weight in the given rule