Discrete and continuous dynamic systems

Continuous time and discrete time nonlinear systems
Nonlinear stability analysis with Lyapunov method

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Feb 2018

Lecture overview

1. Previous notions
2. Nonlinear continuous time state space models
   - Input-affine state space models
   - Linearization
3. Nonlinear stability analysis
   - The Lyapunov method
   - CT-LTI Lyapunov theorem
   - Stability region of nonlinear systems
4. Discrete time nonlinear state space models
   - Discrete event systems
   - Finite automaton
   - Discrete event systems and automata

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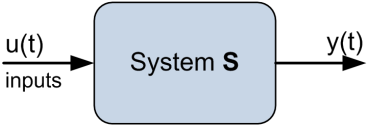
Systems

- System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

- inputs (u) and outputs (y)



CT-LTI state-space models

- **General form - revisited**

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)$$
$$y(t) = Cx(t)$$

  with
  - signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
  - system parameters: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{p \times n}$ ($D = 0$ by using **centering** the inputs and outputs)
- Dynamic system properties:
  - observability
  - controllability
  - stability

DT-LTI state space models

- State space model

$$x(k+1) = \Phi x(k) + \Gamma u(k) \qquad (\textit{state equation})$$
$$y(k) = Cx(k) + Du(k) \qquad (\textit{output equation})$$

- with given initial condition $x(0)$ and

$$x(k) \in \mathbb{R}^n \ , \ y(k) \in \mathbb{R}^p \ , \ u(k) \in \mathbb{R}^r$$

being vectors of finite dimensional spaces and

$$\Phi \in \mathbb{R}^{n \times n} \ , \ \Gamma \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{p \times n} \ , \ D \in \mathbb{R}^{p \times r}$$

being matrices

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General form

Concentrated parameter: (=finite dimensional) general form

$$\dot{x}(t) = \widetilde{f}(x(t), u(t)) \qquad (\textit{state equation})$$
$$y(t) = \widetilde{h}(x(t), u(t)) \qquad (\textit{output equation})$$

with

- the state, input and output vectors x, u and y and
- the smooth nonlinear mappings

$$\widetilde{f} \; : \; \mathbb{R}^n \times \mathbb{R}^r \mapsto \mathbb{R}^n \quad , \quad \widetilde{h} \; : \; \mathbb{R}^n \times \mathbb{R}^r \mapsto \mathbb{R}^p \quad .$$

Input-affine state space models

General form of continuous time nonlinear *input-affine* state-space models

$$\begin{array}{ll} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t)) u_i(t) & (state\ equation) \\ y(t) = h(x(t)) & (output\ equation) \end{array}$$

with

- given initial condition $x(t_0) = x(0)$ and $x(t) \in \mathcal{R}^n$,
- $y(t) \in \mathcal{R}^p$, $u(t) \in \mathcal{R}^r$
- system parameters: smooth nonlinear mappings

$$f \ : \ \mathbb{R}^n \mapsto \mathbb{R}^n \quad , \quad g_i \ : \ \mathbb{R}^n \mapsto \mathbb{R} \quad , \quad h \ : \ \mathbb{R}^n \mapsto \mathbb{R}^p \ .$$

The steady-state point(s)

- Steady-state point: x_0 for a given u_0
- *Input-affine systems*: Solve the steady-state equations with u_0 given

$$0 = f(x_0) + g(x_0)u_0 \qquad (*)$$

$$y_0 = h(x_0)$$

  $(*)$ may have more that one solution (or no solution at all).
- *Centered variables*: $\widetilde{x} = x - x_0$

Linearization

- **Linearization of multivariate functions**:
  $y = h(x_1, \ldots, x_n) \quad , \quad h : \mathcal{R}^n \mapsto \mathcal{R}^m$

$$\widetilde{y} = J^{(h,x)}\Big|_{x_0} \cdot \widetilde{x}$$

$$J_{ji}^{(h,x)} = \frac{\partial h_j}{\partial x_i}$$

  is the Jacobian matrix of f and $y_0 = h(x_0)$

- *Input-affine systems*: Linearize the nonlinear functions in

$$\dot{x} = f(x) + g(x)u = F(x, u)$$

$$y = h(x)$$

  in the neighborhood of the steady-state point (x_0, u_0).

Linearized LTI state-space models

- **Input-affine case**: linearize $y = F(x,u) = f(x) + g(x)u$

$$\begin{aligned}
\widetilde{y} &= J^{(F,x)}\Big|_{x_0,u_0} \cdot \widetilde{x} + J^{(F,u)}\Big|_{x_0,u_0} \cdot \widetilde{u} \\
\widetilde{y} &= \left(J^{(f,x)}\Big|_0 + J^{(g,x)}\Big|_0 u_0) \right) \cdot \widetilde{x} + g(x_0) \cdot \widetilde{u}
\end{aligned}$$

- LTI model form:

$$\begin{aligned}
\dot{\widetilde{x}} &= \widetilde{A}\widetilde{x} + \widetilde{B}\widetilde{u} \\
\widetilde{y} &= \widetilde{C}\widetilde{x} + \widetilde{D}\widetilde{u}
\end{aligned}$$

$$\widetilde{A} = J^{(f,x)}\Big|_0 + J^{(g,x)}\Big|_0 u_0, \quad \widetilde{B} = g(x_0), \quad \widetilde{C} = J^{(h,x)}\Big|_0, \quad \widetilde{D} = 0$$

Linearization

Example

$$\begin{aligned} \dot{x}_1 &= 0.4x_1x_2 - 1.5x_1 \\ \dot{x}_2 &= -0.8x_1x_2 - 1.5x_2 + 1.5u \\ y &= x_2 \end{aligned}$$

Steady-state points with $u_0 = 0$

$$\begin{aligned} 0 &= 0.4x_1x_2 - 1.5x_1 = x_1(0.4x_2 - 1.5) \\ 0 &= -0.8x_1x_2 - 1.5x_2 = x_2(-0.8x_1 - 1.5) \end{aligned}$$

- $x_1 = 0,\ x_2 = 0$
- $x_1 = 1.875,\ x_2 = 3.75$

Linearization

Example (contd)

System parameters and Jacobian matrices

$$f(x) = \begin{bmatrix} 0.4x_1x_2 - 1.5x_1 \\ -0.8x_1x_2 - 1.5x_2 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

$$J^{(f,x)} = \begin{bmatrix} 0.4x_2 - 1.5 & 0.4x_1 \\ -0.8x_2 & -0.8x_1 - 1.5 \end{bmatrix} \quad h(x) = \begin{bmatrix} x_2 \end{bmatrix}$$

Linearized state equation at $x_1 = 0, \; x_2 = 0$

$$\dot{x} = \begin{bmatrix} -1.5 & 0 \\ 0 & -1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} u$$

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Lyapunov theorem of stability

Given an autonomous nonlinear system model with equilibrium point x^*

$$\dot{x} = f(x)$$

- Lyapunov-function: $V : \mathcal{X} \mapsto \mathbb{R}$
  - $V > 0$, if $x \neq x^*$, $V(x^*) = 0$
  - V continuously differentiable
  - V non-increasing, i.e. $\frac{d}{dt} V(x) = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x) \leq 0$

Theorem (Lyapunov stability theorem)

- *If there exists a Lyapunov function to the system* $\dot{x} = f(x)$, $f(x^*) = 0$, *then* x^* *is a stable equilibrium point.*
- *If* $\frac{d}{dt} V < 0$ *then* x^* *is an asymptotically stable equilibrium point.*
- *If the properties of a Lyapunov function hold only in a neighborhood* U *of* x^*, *then* x^* *is a locally (asymptotically) stable equilibrium point.*

A sufficient condition!

Lyapunov theorem – example

- System:

$$\dot{x} = -(x-1)^3$$

- Equilibrium point: $x^* = 1$
- Lyapunov function: $V(x) = (x-1)^2$

$$\frac{d}{dt}V = \frac{\partial V}{\partial x}\dot{x} = 2(x-1)\cdot(-(x-1)^3) =$$
$$= -2(x-1)^4 < 0$$

- The system is **globally asymptotically stable**

CT-LTI Lyapunov theorem – 1

Basic notions:

- $Q \in \mathbb{R}^{n \times n}$ **symmetric matrix**: $Q = Q^T$, i.e. $[Q]_{ij} = [Q]_{ji}$ (every eigenvalue of Q is real)
- symmetric matrix Q is **positive definite (**$Q > 0$**)**: $x^T Q x > 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is positive)
- symmetric matrix Q is **negative definite** $Q < 0$: $x^T Q x < 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is negative)

Theorem (Lyapunov criterion for LTI systems)

The state matrix (A) of an LTI system is a stability matrix if and only if there exists a positive definite symmetric matrix P for every given positive definite symmetric matrix Q such that

$$A^T P + PA = -Q$$

CT-LTI Lyapunov theorem – 2

Proof:

$\Leftarrow$ Assume $\forall\ Q > 0\ \exists\ P > 0$ such that $A^T P + PA = -Q$. Let $V(x) = x^T P x$.

$$\frac{d}{dt} V = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x < 0$$

$\Rightarrow$ Assume A is a stability matrix. Then

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

$$A^T P + PA = \int_{\mathbf{0}}^\infty A^T e^{A^T t} Q e^{At} dt + \int_{\mathbf{0}}^\infty e^{A^T t} Q e^{At} A dt = [e^{A^T t} Q e^{At}]_{\mathbf{0}}^\infty = \mathbf{0} - Q = -Q$$

Quadratic stability region

- Use quadratic Lyapunov function candidate with a given positive definite diagonal weighting matrix Q (tuning parameter!)

$$V\left[x(t)\right] = (x - x^{*})^{T} \cdot Q \cdot (x - x^{*})$$

- Dissipativity condition gives a conservative estimate of the stability region

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}\overline{f}(x)$$

  - $\overline{f}(x) = f(x)$ in the open loop case with $u = 0$
  - $\overline{f}(x) = f(x) + g(x) \cdot C(x)$ in the closed-loop case where $C(x)$ is the static state feedback

Quadratic stability region: an example - 1

- Nonlinear system

$$
\begin{aligned}
\dot{x}_1 &= 0.4 x_1 x_2 - 1.5 x_1 \\
\dot{x}_2 &= -0.8 x_1 x_2 - 1.5 x_2 + 1.5 u \\
y &= x_2
\end{aligned}
$$

- Equilibrium point with $u^* = 7.75$

$$
x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 2 \\ 3.75 \end{bmatrix}
$$

- Locally linearized system

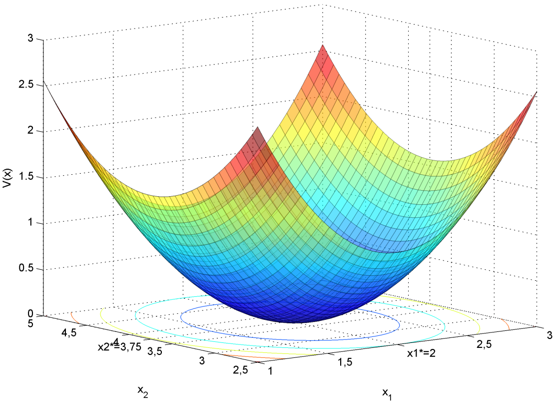
$$
\begin{aligned}
\dot{\tilde{x}} &= \begin{bmatrix} 0 & 0.8 \\ -3 & -3.1 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \tilde{u} \\
\tilde{y} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{x}
\end{aligned}
$$

- Eigenvalues of the state matrix are $\lambda_1 = -1.5$ and $\lambda_2 = -1.6$ so equilibrium x^* (and not the whole system!) is locally asymptotically stable.

Quadratic stability region: an example - 2

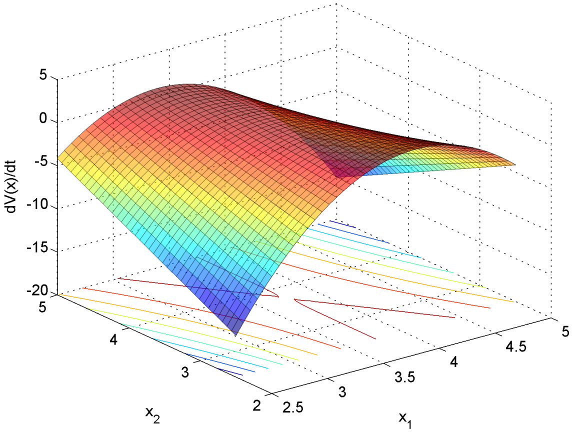
- Quadratic Lyapunov function

$$V(x) = (x - x^*)^T \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot (x - x^*)$$



Quadratic stability region: an example - 2

- Time derivative of the quadratic Lyapunov function



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Discrete time nonlinear state space model

Generalized form of DT-LTI state space model

$$x(k+1) = \Psi(x(k), u(k)) \qquad (\textit{state equation})$$
$$y(k) = h(x(k), u(k)) \qquad (\textit{output equation})$$

Discrete event systems

- Characteristic properties:
  - *discrete valued* signals (for inputs, states, outputs) :
    $x(t) \in \mathbf{X} = \{x_0, x_1, ..., x_n\}$
  - *event*: occurrence of a change in a discrete valued signal
  - *time is discrete*: $T = \{t_0, t_1, ..., t_n\} = \{0, 1, ..., n\}$
- Only the sequence of the events is important
  - sequential and parallel events
  - application area: scheduling, operating procedures, resource allocation

Discrete event systems - discrete time state-space models

Generalization of DT-LTI state-space models

$$x(k+1) = \Psi(x(k), u(k)) \qquad (\textit{state equation})$$
$$y(k) = h(x(k), u(k)) \qquad (\textit{output equation})$$

with given initial conditions $x(0)$, and with nonlinear state Ψ and output h functions.

Discrete event system:

1. non-equividistant sampling (discrete time)
2. discrete valued signals (!!)
3. event: change in the discrete value of a signal

A discrete event system can be described by a special DT state-space model

Finite automaton – abstract model: $\mathbf{A} = (Q, \Sigma, \delta; \Sigma_O, \varphi)$

- Set of states: Q
- Finite alphabet of the input tape: $\Sigma = \{\#; a, b, ...\}$
- State-transition function: $\delta : Q \times \Sigma \to Q$
- *Initial and final state(s)*: $Q_I, \; Q_F \; \subseteq \; Q$
- Finite alphabet of the output tape: $\Sigma_O = \{\#; \alpha, \beta, ...\}$
- Output function: $\varphi : Q \to \Sigma_O$

Graphical description: using a weighted directed graph

- Vertices: states (Q)
- Edges: state transitions (δ)
- Edge weights: input symbols (Σ)

A discrete event system can be modelled by a finite automaton

Automata and discrete event systems

	Automata model	Discrete event SS model
State space	Q	$\mathcal{X} \in \mathbb{Z}^n$
Input u	string from Σ	discrete valued discrete time signal
Output y	string from Σ_O	discrete valued discrete time signal
State equation	$q(k+1) = \delta(q(k), u(k))$	$x(k+1) = \Psi(x(k), u(k))$
Output equation	$y(k) = \varphi(x(k))$	$y(k) = h(x(k), u(k))$