Discrete and continuous dynamic systems

Bounded input bounded output (BIBO) and asymptotic stability
Continuous and discrete time linear time-invariant systems

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Lecture overview

1. Previous notions
   - CT-LTI state space models
   - DT-LTI state space models
   - Poles
2. The notion of stability
   - Signal norms
3. Bounded input-bounded output (BIBO) stability - continuous time systems
   - BIBO stability for SISO CT-LTI systems
4. Asymptotic stability - continuous time systems
   - The notion of asymptotic stability
   - Motivating example
   - Asymptotic stability of CT-LTI systems
5. Discrete time stability
   - Stability of DT systems
   - Stability of DT-LTI systems

Overview

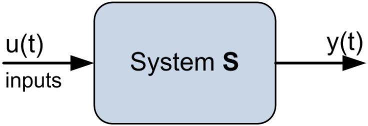
1. Previous notions
   - CT-LTI state space models
   - DT-LTI state space models
   - Poles
2. The notion of stability
3. Bounded input-bounded output (BIBO) stability - continuous time systems
4. Asymptotic stability - continuous time systems
5. Discrete time stability

Systems

- System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

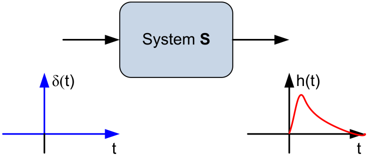
- inputs (u) and outputs (y)



CT-LTI I/O system models

- Time domain:Impulse response function
  is the response of a SISO LTI system to a Dirac-delta input function with zero initial condition.
- The output of **S** can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$



CT-LTI state-space models

- **General form - revisited**

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)$$
$$y(t) = Cx(t)$$

with

  - signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
  - system parameters: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ ($D = 0$ by using **centering** the inputs and outputs)

- Dynamic system properties:
  - observability
  - controllability
  - stability

DT-LTI state space models

- State space model

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \qquad (\textit{state equation}) \\
y(k) &= Cx(k) + Du(k) \qquad (\textit{output equation})
\end{aligned}
$$

- with given initial condition $x(0)$ and

$$x(k) \in \mathbb{R}^n \ , \ y(k) \in \mathbb{R}^p \ , \ u(k) \in \mathbb{R}^r$$

being vectors of finite dimensional spaces and

$$\Phi \in \mathbb{R}^{n \times n} \ , \ \Gamma \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{p \times n} \ , \ D \in \mathbb{R}^{p \times r}$$

being matrices

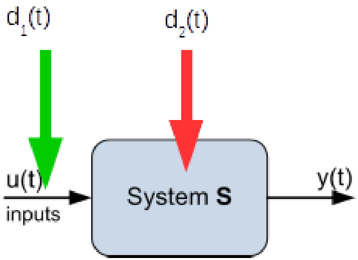
Poles of CT-LTI and DT-LTI systems

	continuous time system	discrete time system
state eq.	$\dot{x}(t) = Ax(t) + Bu(t)$	$x(kh+h) = \Phi x(kh) + \Gamma u(kh)$ $\Phi = e^{Ah}$
output eq.	$y(t) = Cx(t)$	$y(kh) = Cx(kh)$
poles	$\lambda_i(A)$	$\lambda_i(\Phi)$ $\lambda_i(\Phi) = e^{\lambda_i(A)h}$

Overview

Stability

Stability expresses the resistance of a system against disturbances.



System response to two kinds of disturbances

- Small persistent disturbance on input(s) (d_1): ***external or bounded input bounded output (BIBO) stability***
- Impulse type effect on the state moving it out of steady state (d_2): ***internal or asymptotic stability***

Scalar valued signals

- vector norms: $v \in \mathbb{R}^n$

$$||v||_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \quad , \quad ||v||_1 = \sum_{i=1}^{n} |v_i| \quad , \quad ||v||_\infty = \max |v_i|$$

- discrete time signal: $f(k) \in \mathbb{R}, \ \forall k \geq 0$

$$\text{norm: } ||f||_q = \left(\sum_{0}^{\infty} |f(k)|_\nu^q \right)^{\frac{1}{q}}$$

- continuous time signal $f(t) \in \mathbb{R}, \ \forall t \geq 0$

$$\text{norm: } ||f||_q = \left(\int_0^\infty |f(t)|_\nu^q \right)^{\frac{1}{q}}$$

Vector valued signals

- continuous time signal: $f(t) \in \mathbb{R}^n, \ \forall t \geq 0$
- $|| \cdot ||_n$ is a norm in $\mathbb{R}^n$ (e.g. Euclidean)

$$L_q(\nu) = \left\{ f : \mathbb{R}_0^+ \mapsto \mathbb{R}^n \mid f \text{ is measurable and } \int_0^\infty ||f(t)||_\nu^q < \infty \right\}$$

$$\text{norm: } ||f||_q = \left(\int_0^\infty ||f(t)||_\nu^q \right)^{\frac{1}{q}}$$

- Remark: The case L_2 is special, because the norm comes from an inner product (L_2 is a Hilbert-space)

Overview

BIBO stability – general

Definition (BIBO stability)

A system is *externally or BIBO stable* if for any bounded input it responds with a bounded output

$$||u|| \leq M_1 < \infty \Rightarrow ||y|| \leq M_2 < \infty$$

where $||.||$ is a signal norm.

- This applies to any type of systems.
- Stability is a system property, i.e. it is realization-independent.

BIBO stability – 1

- Bounded input-bounded output (BIBO) stability for SISO systems

$$|u(t)| \leq M_1 < \infty, \ \forall t \in [0, \infty[\quad \Rightarrow \quad |y(t)| \leq M_2 < \infty, \ \forall t \in [0, \infty[$$

Theorem (BIBO stability)

A SISO LTI system is BIBO stable if and only if

$$\int_0^\infty |h(t)| dt \leq M < \infty$$

where $M \in \mathbb{R}^+$ and h is the impulse response function.

BIBO stability – 2

Proof:

$\Leftarrow$ Assume $\int_0^\infty |h(t)|dt \leq M < \infty$ and u is bounded, i.e. $|u(t)| \leq M_1 < \infty, \forall t \in \mathbb{R}_0^+$. Then

$$|y(t)| \leq |\int_0^\infty h(\tau)u(t-\tau)d\tau| \leq M_1 \int_0^\infty |h(\tau)|d\tau \leq M_1 \cdot M = M_2$$

$\Rightarrow$ (indirect) Assume $\int_0^\infty |h(\tau)|d\tau = \infty$, but the system is BIBO stable. Consider the **bounded** input:

$$u(t-\tau) = \text{sign } h(\tau) = \begin{cases} 1 & \text{if} \quad h(\tau) > 0 \\ 0 & \text{if} \quad h(\tau) = 0 \\ -1 & \text{if} \quad h(\tau) < 0 \end{cases}$$

Overview

Asymptotic stability – general

Definition ((local) asymptotic stability)

An equilibrium/steady-state point x^* of truncated/autonomous system with state equation

$$\dot{x}(t) = F(x(t)) \quad , \quad x(0) = x_0 \ (\neq x^*) \quad , \quad F(x^*) = 0$$

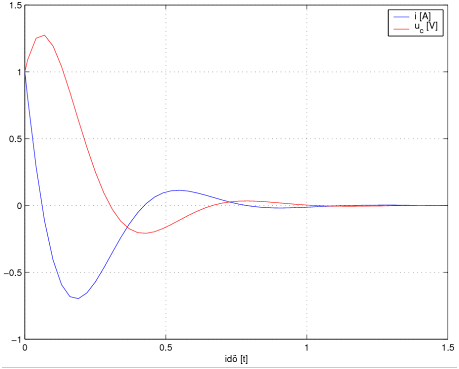
is *internally or asymptotically stable* if for any initial state $x_0 \neq x^*$ (from a neighbourhood of G_{x^*} of x^*)

$$\lim_{t \to \infty} x(t) = x^*$$

- This applies to any type of continuous time systems.
- For discrete time systems a similar definition is applicable with $x(k+1) = F(x(k))$.

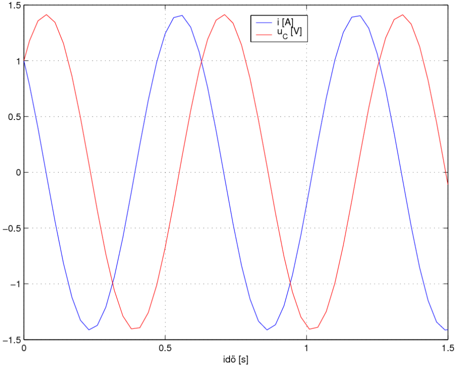
Example: asymptotic stability

RLC circuit, parameters: $R = 1\ \Omega,\ L = 10^{-1} H,\ C = 10^{-1} F.$
$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V



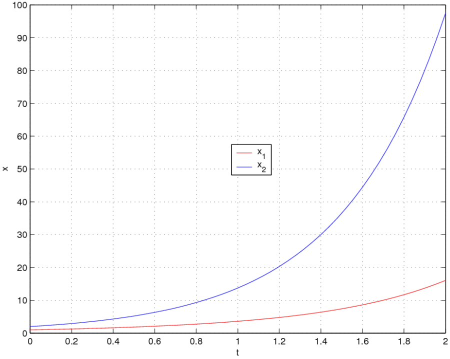
Non-asymptotic stability

(R)LC circuit, parameters: $R = 0\ \Omega(!)$, $L = 10^{-1} H$, $C = 10^{-1} F$.
$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V



Example: instability

$$\begin{array}{rcl} \dot{x}_1 & = & x_1 + 0.1x_2 \\ \dot{x}_2 & = & -0.2x_1 + 2x_2 \end{array} , \quad x(0) = [1 \quad 2]^T$$



Stability of CT-LTI systems

- (Truncated) LTI state equation with ($u \equiv 0$):

$$\dot{x} = A \cdot x, \quad x \in \mathbb{R}^n, \ A \in \mathbb{R}^{n \times n}, \ x(0) = x_0$$

- Equilibrium pont: $x^* = 0$
- Solution:

$$x(t) = e^{At} \cdot x_0$$

- Recall: A diagonalizable (there exists invertible T, such that

$$T \cdot A \cdot T^{-1}$$

is diagonal) if and only if, A has n linearly independent eigenvectors.

Asymptotic stability of LTI systems – 1

Stability types:

- the real part of every eigenvalue of A is negative (A is a *stability matrix*): **asymptotic stability**
- A has eigenvalues with zero and negative real parts
  - the eigenvectors related to the zero real part eigenvalues are linearly independent: **(non-asymptotic) stability**
  - the eigenvectors related to the zero real part eigenvalues are not linearly independent: **(polynomial) instability**
- A has (at least) an eigenvalue with positive real part: **(exponential) instability**

Asymptotic stability of LTI systems – 2

Theorem

The eigenvalues of a square $A \in \mathcal{R}^{nxn}$ matrix remain unchanged after a similarity transformation on A by a transformation matrix T:

$$A' = TAT^{-1}$$

Proof:

Let us start with the eigenvalue equation for matrix A

$$A\xi = \lambda\xi \ , \ \xi \in \mathcal{R}^n \ , \ \lambda \in \mathbb{C}$$

If we transform it using $\xi' = T\xi$ then we obtain

$$TAT^{-1}T\xi = \lambda T\xi$$

$$A'\xi' = \lambda\xi'$$

Asymptotic stability of LTI systems – 3

Theorem

A CT-LTI system is asymptotically stable iff A is a stability matrix.

Sketch of *Proof*: Assume A is diagonalizable

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

$$\bar{x}(t) = e^{\bar{A}t} \cdot \bar{x}_0 \quad , \quad e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}$$

BIBO and asymptotic stability

Theorem

Asymptotic stability implies BIBO stability for LTI systems.

Proof:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad y(t) = Cx(t)$$

$$\begin{aligned} ||x(t)|| &\leq ||e^{At}x(t_0) + M\int_0^t e^{A(t-\tau)}Bd\tau|| = \\ &= ||e^{At}(x(t_0) + M\int_0^t e^{-A\tau}Bd\tau)|| = \\ &= ||e^{At}(x(t_0) + M[-A^{-1}e^{-A\tau}B]_0^t)|| = \\ &= ||e^{At}[x(t_0) - MA^{-1}e^{-At}B + MA^{-1}B]|| \end{aligned}$$

$$||x(t)|| \leq ||e^{At}(x(t_0) + MA^{-1}B) - MA^{-1}B||$$

BIBO stability does not necessarily imply asymptotic stability.

Overview

Stability of discrete time systems – 1

- Truncated state equation

$$x(k+1) = f(x(k), k)$$

  with a ordinary solution $x^0(k)$ for $x^0(k_0)$ and a *perturbed solution* $x(k)$ for $x(k_0)$.

- Stability of a solution $x^0(k)$ is stable if for a given $\epsilon > 0$ there exists a $\delta(\epsilon, k_0)$ such that all solutions with $||x(k_0) - x^0(k_0)|| < \delta$ fulfill $||x(k) - x^0(k)|| < \epsilon$ for all $k \geq k_0$.

- Asymptotic stability $x^0(k)$ is asymptotically stable if it is stable and $||x(k) - x^0(k)|| \to 0$ when $k \to \infty$ provided that $||x(k_0) - x^0(k_0)||$ is small enough.

Stability of discrete time systems – 2

- BIBO stability

A discrete time system is externally or BIBO stable if for any

$$||u|| \leq M_1 < \infty \Rightarrow ||y|| \leq M_2 < \infty$$

where $||.||$ is a suitable *signal norm*.

Stability of DT-LTI systems – 1

- Consider a truncated state equation with $u(k) = 0, \; k = 0, 1, 2, \ldots$

$$x(k+1) = \Phi x(k)$$

  - $x^0(k)$ for $x^0(0) = a^0$ as the ordinary solution and
  - $x(k)$ for $x(0) = a$ as a "perturbed solution".
- The difference $\overline{x} = x - x^0$ satisfies

$$\overline{x}(k+1) = \Phi \overline{x}(k) \quad , \quad \overline{x}(0) = a - a^0$$

$\Rightarrow$ Stability is a system property for LTI systems

Stability of DT-LTI systems – 2

- Solution of the truncated state equation $x(k+1) = \Phi x(k),\ x(0) = x_0$

$$x(k) = \Phi^k x(0)$$

- Bring the matrix Φ^k into diagonal form and use that its eigenvalues $\lambda_i(\Phi^k) = \lambda_i(\Phi)^k$ thus

$$x(k) \longrightarrow 0 \quad \Longleftrightarrow \quad |\lambda_i(\Phi)| < 1$$

Theorem

A DT-LTI system is asymptotically stable if and only if $\lambda_i(\Phi)$ are strictly inside the unit disc.

Theorem

Asymptotic stability implies BIBO stability.