Coloured Petri Nets

Modelling and Validation of Concurrent Systems

Chapter 4: Formal Definition of CP-nets

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Syntax

CPN = (P, T, A, Σ, V, C, G, E, I)

Semantics

$$\sum_{MS\ (t,b)\in Y}^{++} E(p,t)\langle b\rangle \ll= M(p) \text{ for all } p \in P$$

Why do we need a formal definition?

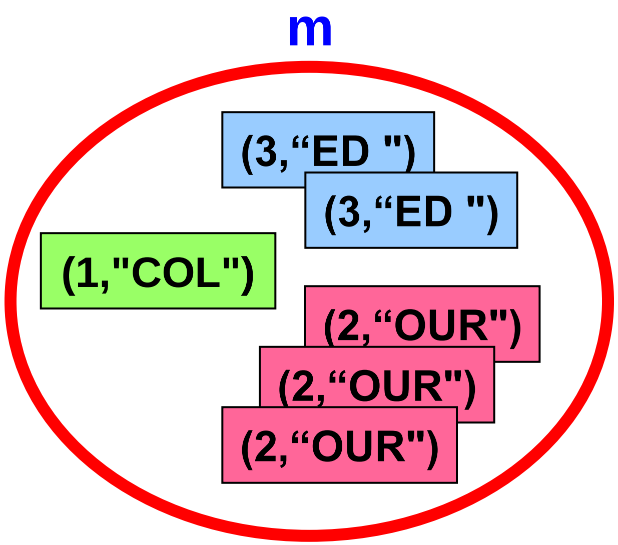
- The formal definition is unambiguous.
- It provides a more precise and complete description than an informal explanation.

- Users who are satisfied with the informal explanation can skip the formal definition.
- Only few programmers know the formal definition of the programming language they are using.

- We define:
  - Multisets.
  - Syntax of Coloured Petri Nets.
  - Semantics of Coloured Petri Nets.

Multiset

- Similar to a set but with multiple occurrences of elements.



Elements in multiset

Non-negative integers

- **Function** **NOxDATA** $\rightarrow \mathbb{N}$**:**

$$m(s) = \begin{cases} 1 & \text{if } s = (1,\text{"COL"}) \\ 3 & \text{if } s = (2,\text{"OUR"}) \\ 2 & \text{if } s = (3,\text{"ED "}) \\ 0 & \text{otherwise} \end{cases}$$

- **Sum:** m = 1‘(1,"COL") ++ 3‘(2,"OUR") ++ 2‘(3,"ED ")

Number of appearances (coefficient)

Elements (from NOxDATA)

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Formal definition of multisets

- Let $S = \{s_1, s_2, s_3, \ldots\}$ be a non-empty set.

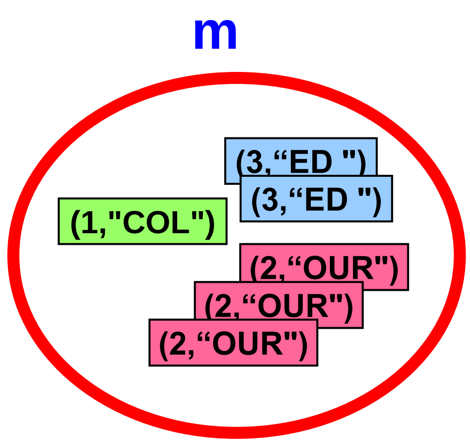
- A multiset over S is a function $m : S \rightarrow \mathbb{N}$ mapping each element $s \in S$ into a non-negative integer $m(s) \in \mathbb{N}$ called the number of appearances (or coefficient) of s in m.

- A multiset m is also written as a sum:

$$\sum_{s \in S}^{++} m(s)`s = m(s_1)`s_1 ++ m(s_2)`s_2 ++ m(s_3)`s_3 ++ m(s_4)`s_4 ++ \ldots$$

- Notation:
  - S_{MS} is the set of all multisets over S.
  - $\emptyset_{MS}$ is the empty multiset (polymorphic).

Membership of multiset



- (1,”COL”), (2,”OUR”) and (3,”ED ”) are members of the multiset m.
- (4,”PET”) and (17,”CPN”) are not members.

$$\forall s \in S: s \in m \Leftrightarrow m(s) > 0.$$

Membership of multiset

Comparison of integers

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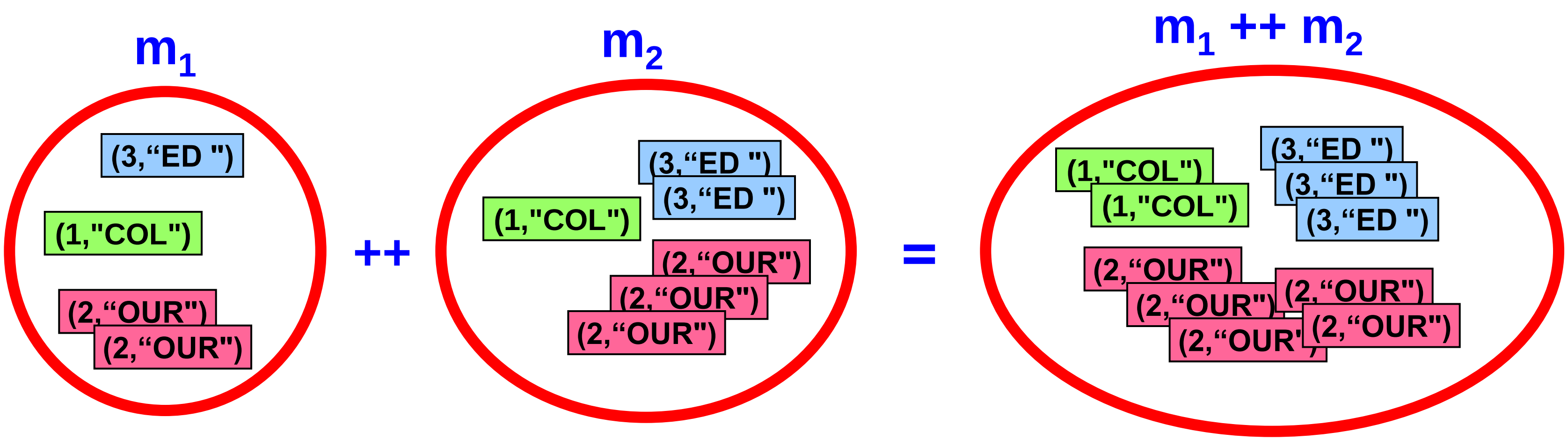
Addition of multisets

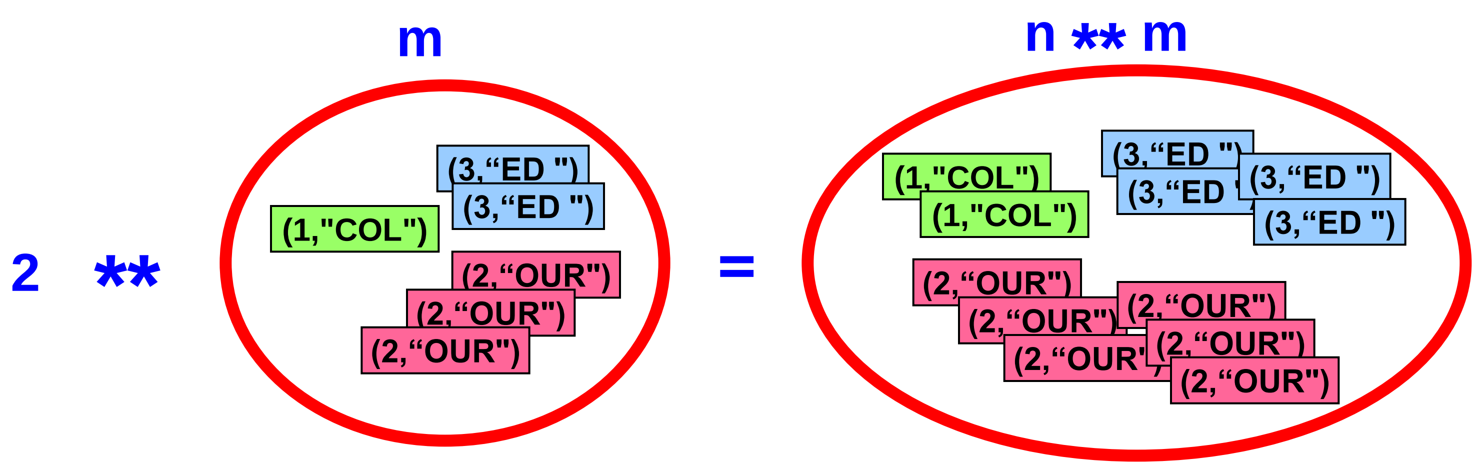


$$\forall s \in S: (m_1 ++ m_2)(s) = m_1(s) + m_2(s).$$

Addition of multisets

Addition of integers

Scalar multiplication of multisets

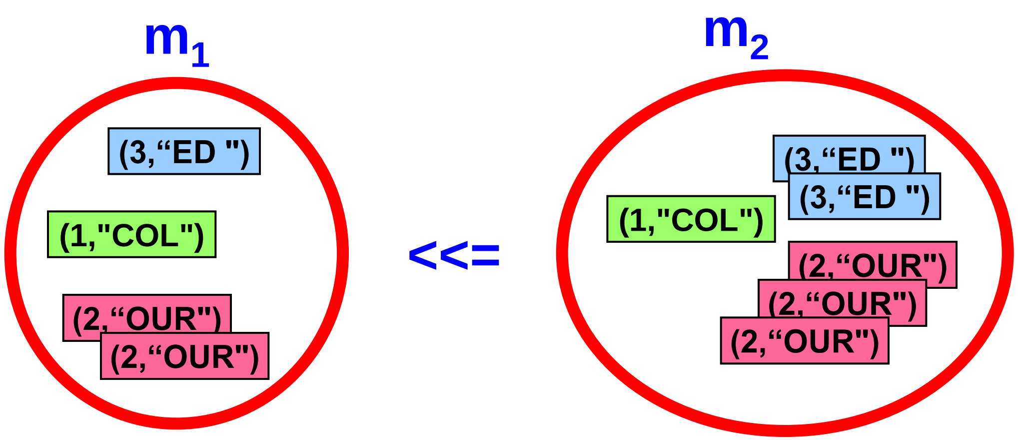


$$\forall s \in S: (n ** m)(s) = n * m(s).$$

Scalar multiplication of multiset

Multiplication of integers

Comparison of multisets

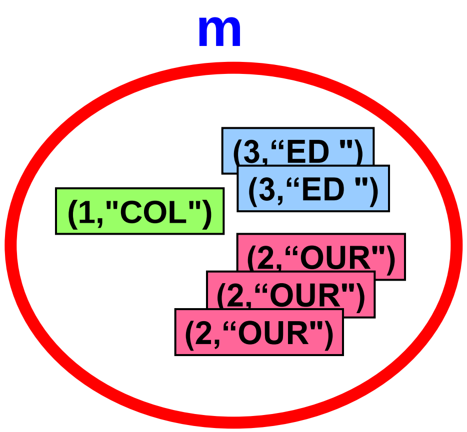


$$m_1 \ll= m_2 \quad \Leftrightarrow \quad \forall s \in S: m_1(s) \leq m_2(s).$$

Smaller than or equal for multisets

Smaller than or equal for integers

Size of multiset



- This multiset contains six elements.

$$|m| = \sum_{s \in S} m(s).$$

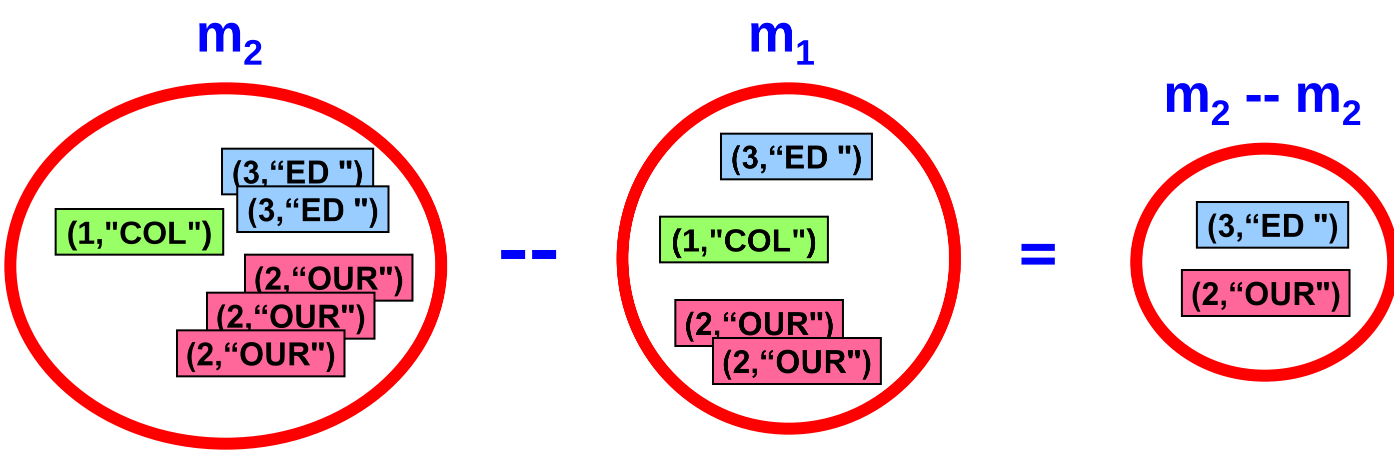
Size of multiset

Summation of integers

When $|m| = \infty$ we say that m is infinite.

Subtraction of multisets

- When m_1 <<= m_2 we also define subtraction:



$$\forall s \in S: (m_2 -\!- m_1)(s) = m_2(s) - m_1(s).$$

Subtraction of multisets

Subtraction of integers

The condition m_1 <<= m_2 ensures that $m_2(s) - m_1(s)$ is non-negative, i.e., that $m_2 -\!- m_1$ is a multi-set

Formal definition of Coloured Petri Nets

A Coloured Petri Net is a nine-tuple CPN = (P, T, A, Σ, V, C, G, E, I).

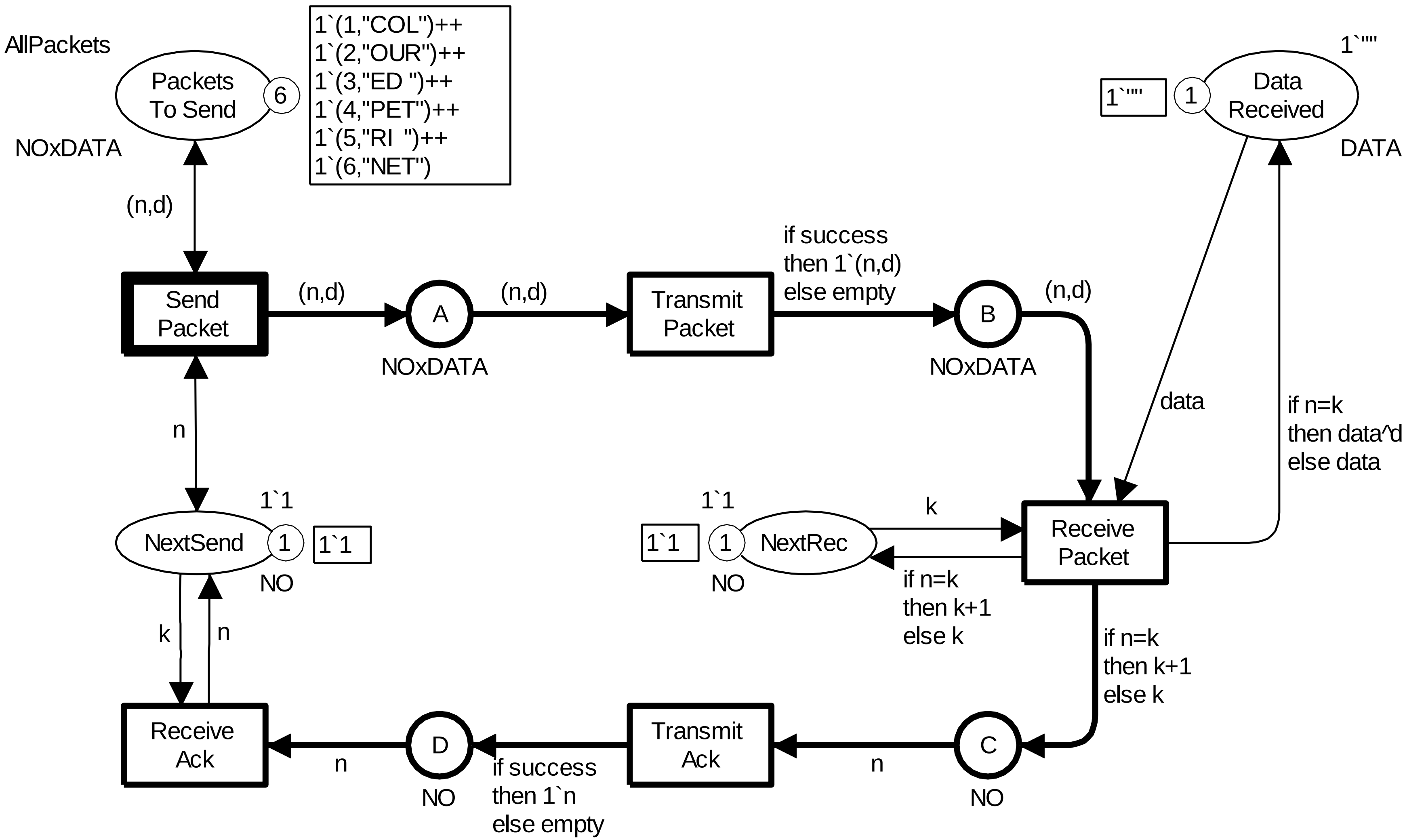
- P set of places.
- T set of transitions.
- A set of arcs.
- Σ set of colour sets.
- V set of variables.
- C colour set function (assigns colour sets to places).
- G guard function (assigns guards to transitions).
- E arc expression function (assigns arc expressions to arcs).
- I initialisation function (assigns initial markings to places).

Net structure

Types and variables

Net inscriptions

Example to illustrate the formal definitions



Places and transitions

- A finite set of places P.

  P = { PacketsToSend, A, B, DataReceived, NextRec, C,D, NextSend }.

- A finite set of transitions T.
- We demand that $P \cap T = \emptyset$.

A node is either a place or a transition – it cannot be both

T = { SendPacket,TransmitPacket, ReceivePacket,TransmitAck, ReceiveAck }.

Arcs

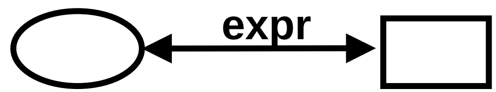
- A set of directed arcs A.
- We demand that $A \subseteq P \times T \cup T \times P$.

Each arc starts in a place and ends in a transition – or it starts in a transition and ends in a place

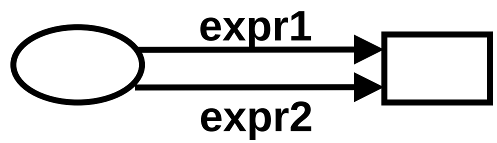
A = { (PacketsToSend, SendPacket), (SendPacket, PacketsToSend),
(SendPacket, A), (A,TransmitPacket), (TransmitPacket, B),
(B, ReceivePacket), (NextRec, ReceivePacket), (ReceivePacket, NextRec),
(DataReceived, ReceivePacket), (ReceivePacket, DataReceived),
(ReceivePacket, C), (C, TransmitAck), (TransmitAck, D), (D, ReceiveAck),
(ReceiveAck, NextSend), (NextSend, ReceiveAck),
(NextSend, SendPacket), (SendPacket, NextSend) }.

Arcs

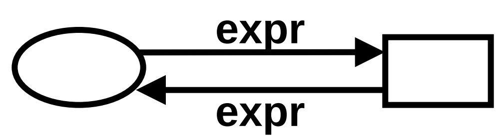
- In the formal definition we do not have:
  - double-headed arcs:



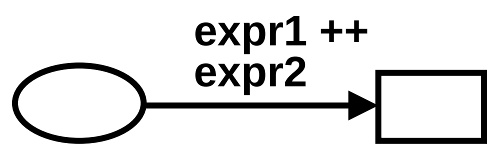
  - parallel arcs:



- CPN Tools allow these and consider them to be shorthands for:
  - two oppositely directed arcs with the same arc expression:



  - addition of the two arc expressions:



Colour sets and variables

- A finite set of non-empty colour sets Σ.

  Σ = { NO, DATA, NOxDATA, BOOL }.

- A finite set of typed variables V.

  We demand that $Type[v] \in \Sigma$ for all $v \in V$.

  Type of variable must be one of those that is defined in Σ

  V = { n : NO, k : NO, d : DATA, data : DATA, success : BOOL }.

Colour sets for places

- A colour set function $C : P \rightarrow \Sigma$.

  Assigns a colour set to each place.

$$C(p) = \begin{cases} \text{NO} & \text{if } p \in \{\text{NextSend, NextRec, C, D}\} \\ \text{DATA} & \text{if } p = \text{DataReceived} \\ \text{NOxDATA} & \text{if } p \in \{\text{PacketsToSend, A, B}\} \end{cases}$$

Guard expressions

All variables must belong to V

- A guard function $G : T \rightarrow EXPR_V$.

  Assigns a guard to each transition.

  We demand that $Type[G(t)] = Bool$ for all $t \in T$.

  G(t) = true for all $t \in T$.

The guard expression must evaluate to a boolean

- In the formal definition we demand all transitions to have a guard.
- CPN Tools consider a missing guard to be a shorthand for the guard expression true which always evaluates to true.
- Hence we have omitted all guards in the protocol example.

Guard expressions

- CPN Tools consider a list of Boolean expressions:

  $$[expr_1, expr_2, \ldots, expr_n]$$

  to be a shorthand for:

  $$expr_1 \wedge expr_2 \wedge \ldots \wedge expr_n$$

- We recommend to write all guards as a list even when they only have a single Boolean expression:

  [expr]

- In this way it is easy to distinguish guards from other kinds of net inscriptions.

Arc expressions

- An arc expression function $E : A \rightarrow EXPR_V$.

  Assigns an arc expression to each arc.

  We demand that $Type[E(a)] = C(p)_{MS}$ for all $a \in A$, where p is the place connected to the arc a.

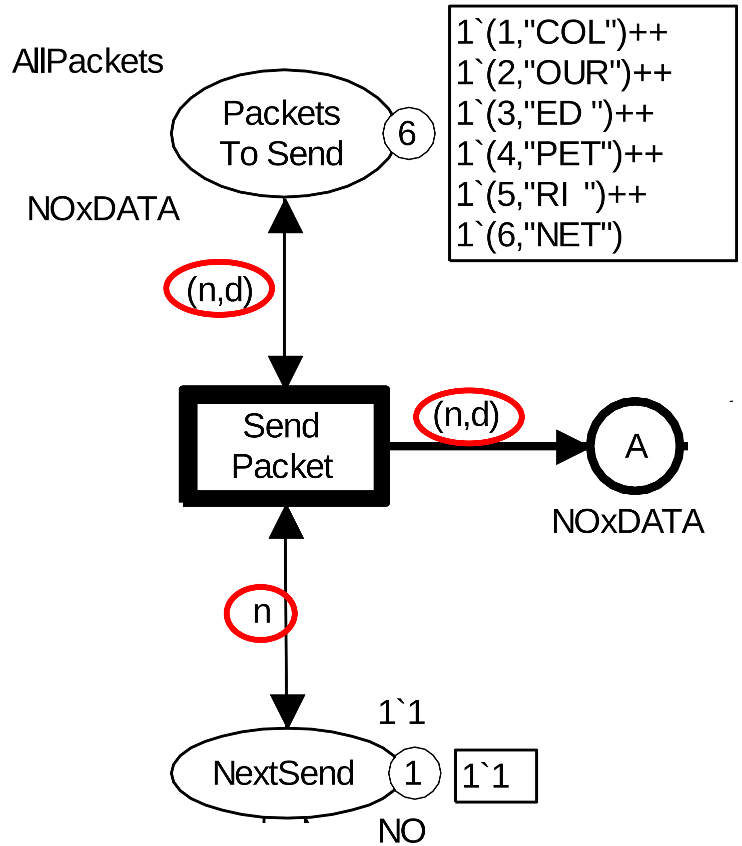
All variables must belong to V

Arc expression must evaluate to a multiset of tokens belonging to the colour set of the connected place

$$\mathbf{E(a)} = \begin{cases} \text{1`(n,d)} & \text{if } a \in \{\, (\text{PacketsToSend}, \text{SendPacket}), \ldots \,\} \\ \text{1`n} & \text{if } a \in \{\, (\text{C}, \text{TransmitAck}), (\text{D}, \text{ReceiveAck}) \ldots \,\} \\ \text{1`data} & \text{if } a = (\text{DataReceived}, \text{ReceivePacket}) \,\} \\ \ldots\ldots\ldots & \end{cases}$$

Arc expressions

- In the formal definition we demand all arc expressions to evaluate to multisets.
- CPN Tools consider an arc expression expr which evaluates to a single value to be a shorthand for 1`expr.
- Hence we can write n and (n,d) instead of 1`n and 1`(n,d).



Initialisation expressions

- An initialisation function $I : P \rightarrow EXPR_{\emptyset}$.
  Assigns an initial marking to each place.
  We demand that
  - $Type[I(p)] = C(p)_{MS}$ for all $p \in P$.

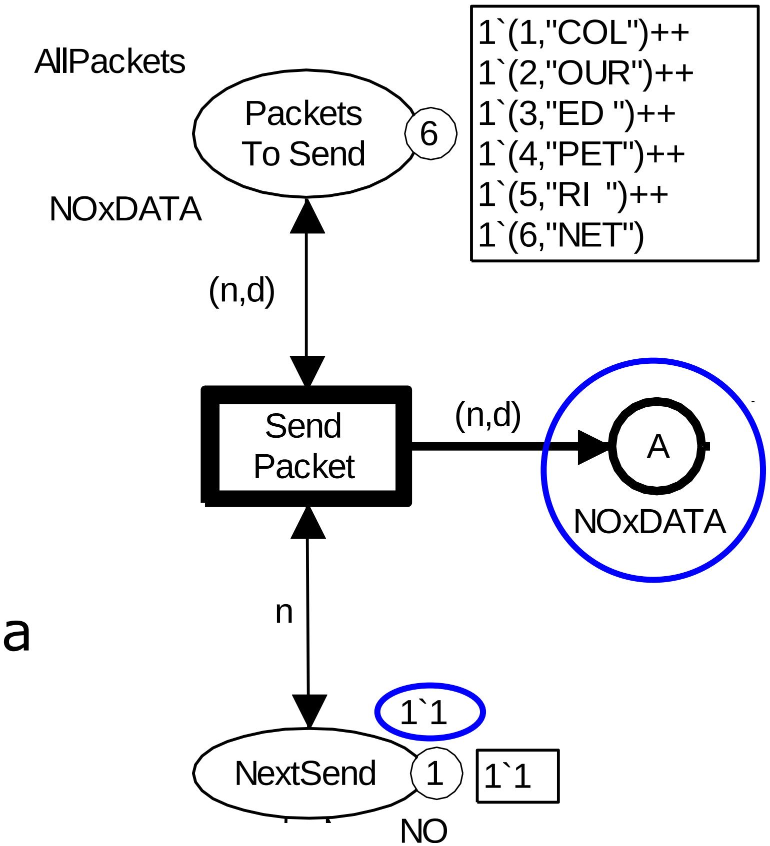
Initialisation expression is not allowed to contain any variables

Initialisation expression must evaluate to a multiset of tokens belonging to the colour set of the place

$$
\mathbf{I(p)} = \begin{cases} \text{AllPackets} & \text{if } p = \text{PacketsToSend} \\ 1`1 & \text{if } p \in \{ \text{NextSend, NextRec} \} \\ 1`\text{""} & \text{if } p = \{ \text{DataReceived} \} \\ \emptyset_{MS} & \text{otherwise} \end{cases}
$$

Initialisation expressions

- In the formal definition we demand all places to have an initialisation expression and that these evaluate to multisets.
- CPN Tools consider a missing initialisation expression to be a shorthand for $\emptyset_{MS}$.
- Hence we are allowed to omit the initialisation expression for place A.
- CPN Tools consider an initialisation expression expr which evaluates to a single value to be a shorthand for 1`expr.
- Hence we could have written 1 instead of 1`1.



Questions about CPN syntax

A. Can a node be both a place and a transition?
B. Can we have an infinite number of places?
C. Can we have an arc from a place to another place?
D. Can a transition have two guards?
E. Can a guard evaluate to an integer?
F. Can an arc expression evaluate to a multiset of booleans?
G. Can we have a variable in an initial marking expression?
H. Can an arc expression always evaluate to empty?

Find those where the answer is YES?

Markings

- A marking is a function M mapping each place p into a multiset of tokens $M(p) \in C(p)_{MS}$.

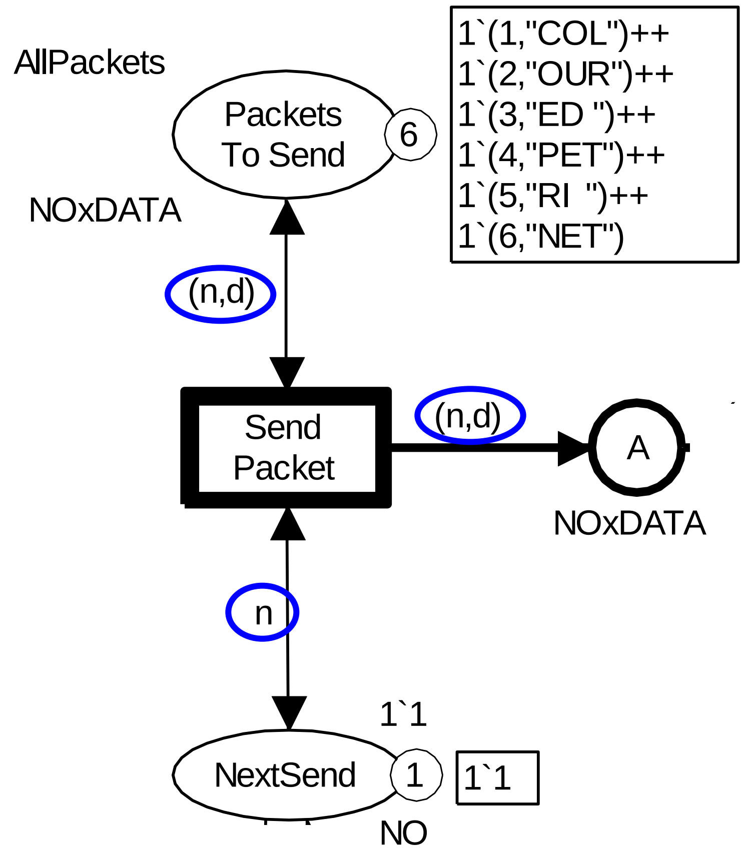
All token values must belong to the colour set of the place

- The initial marking M_0 is defined by $M_0(p) = I(p)<>$ for all $p \in P$.

Initialisation expression has no variables. Hence it is evaluated in the empty binding

Variables of a transition

- The variables of a transition are those that appear in the guard or in an arc expression of an arc connected to the transition.
- The set of variables is denoted $Var(t) \subseteq V$.
- Var(SendPacket) = {n,d}.



Bindings and binding elements

- A binding of a transition t is a function b mapping each variable $v \in Var(t)$ into a value $b(v) \in Type[v]$.

Each variable must be bound to a value in its type

- Bindings are written in: brackets: <n=1,d="COL">.
- The set of all bindings for a transition t is denoted B(t).

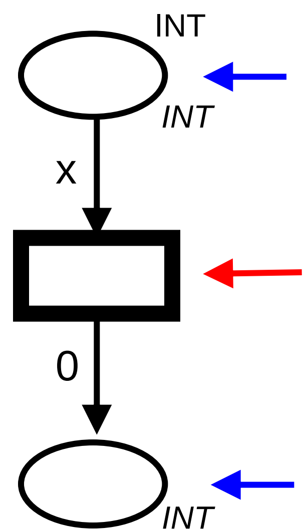
Transition

Binding for t

- A binding element is a pair (t,b) such that t is a transition and $b \in B(t)$.
- The set of all binding elements of a transition t is denoted BE(t).
- The set of all binding elements in CPN is denoted BE.

Steps

- A step $Y \in BE_{MS}$ is a non-empty and finite multiset of binding elements.
- Why forbid empty steps?
  - We would have steps with no effect.
  - It would be impossible to reach a dead marking, i.e., a marking without enabled steps.
- Why forbid infinite steps?
  - We would be able to produce markings which are not multisets.



Infinitely many tokens

Enabled concurrently with itself infinitely many times

Infinitely many tokens with value zero

This is illegal. Multisets cannot have infinite coefficients

Evaluation of guards and arc expressions

- The rules for enabling and occurrence are based on evaluation of guards and arc expressions.

G(t)<b>	**Evaluation of the guard expression for t in the binding b**
E(a)<b>	**Evaluation of the arc expression for a in the binding b**
E(p,t)<b>	**Evaluation of the arc expression on the arc from p to t in the binding b.** **If no such arc exists E(p,t) = $\emptyset_{MS}$**
E(t,p)<b>	**Evaluation of the arc expression on the arc from t to p in the binding b.** **If no such arc exists E(t,p) = $\emptyset_{MS}$**

Enabling of single binding element

- A binding element (t,b)∈BE is enabled in a marking M if and only if the following two properties are satisfied:

  - G(t)<b> = true.

    Guard must evaluate to true

  - E(p,t)<b> <<= M(p)   for all p∈P.

    Smaller than or equal for multisets

    The tokens demanded by the input arc expressions must be present in the marking M

Occurrence of single binding element

- When the binding element $(t,b) \in BE$ is enabled in a marking M, it may occur leading to a new marking M' defined by:

  - $M'(p) = (M(p) \,{-}{-}\, E(p,t)\langle b\rangle) \,{+}{+}\, E(t,p)\langle b\rangle$ for all $p \in P$.

New marking

Old marking

Subtract tokens consumed by input arcs

Add tokens produced by output arcs

Enabling of step

- A step $Y \subseteq BE_{MS}$ is enabled in a marking M if and only if the following two properties are satisfied:
  - $G(t)\langle b\rangle = \text{true}$ for all $(t,b) \in Y$. **All guards must evaluate to true**
  - $\underset{(t,b)\in Y}{{}_{MS}\sum^{++}} E(p,t)\langle b\rangle \ll= M(p)$ for all $p \in P$. **The tokens demanded by the input arc expressions must be present in the marking M**

Summation over a multiset Y

Smaller than or equal for multisets

Occurrence of step

- When the step $Y \in BE_{MS}$ is enabled in a marking M, it may occur leading to a new marking M′ defined by:
  - $M'(p) = (M(p) \;{-}{-}\; \sum_{MS\,(t,b)\in Y}^{++} E(p,t)\langle b\rangle) \;{+}{+}\; \sum_{MS\,(t,b)\in Y}^{++} E(t,p)\langle b\rangle$ for all $p \in P$.

New marking

Old marking

Subtract tokens consumed by input arcs

Add tokens produced by output arcs

Notation for occurrence and enabling

$M_1 \xrightarrow{Y} M_2$ **Step Y occurs in marking M_1 leading to marking M_2**

$M_1 \longrightarrow M_2$ **Marking M_2 can be reached from marking M_1 (by the occurrence of an unknown step)**

$M_1 \xrightarrow{Y}$ **Step Y is enabled in marking M_1 (leading to an unknown marking)**

Finite occurrence sequence

$$M_1 \xrightarrow{Y_1} M_2 \xrightarrow{Y_2} M_3 \;\; \ldots\ldots \;\; M_n \xrightarrow{Y_n} M_{n+1}$$

- Length $n \geq 0$.
- All markings in the sequence are reachable from M_1.
- An arbitrary marking is reachable from itself by the trivial occurrence sequence of length 0.

Infinite occurrence sequence

$$M_1 \xrightarrow{Y_1} M_2 \xrightarrow{Y_2} M_3 \xrightarrow{Y_3} \ldots\ldots$$

- $\mathfrak{R}(M)$ **The set of markings reachable from M**
- $\mathfrak{R}(M_0)$ **The set of reachable markings**

Diamond property

Y_1 **can occur followed by** Y_2

$$\begin{tikzcd} & M'' \arrow[dr, "Y_2"] & \\ M \arrow[ur, "Y_1"] \arrow[rr, "Y"] \arrow[dr, "Y_2"'] & & M' \\ & M''' \arrow[ur, "Y_1"'] & \end{tikzcd}$$

Y_2 **can occur followed by** Y_1

Assume that
Y can be divided into
two substeps:

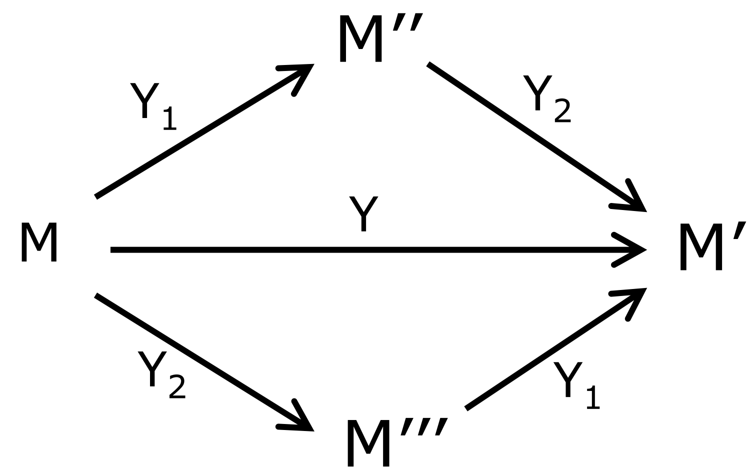
$$Y = Y_1 ++ Y_2$$

In all three cases we
reach the same marking

- This is called the diamond property.
- It can be proved from the definition of enabling and occurrence.
- It plays an important role in Petri net theory.

Diamond property

- The diamond property follows from the fact that the effect of a step is independent of the marking in which it occurs.
- The diamond property is not satisfied by ordinary programming languages.




```
x := x+1;
x := 0;
```

```
x := 0;
x := x+1;
```

- Repeated use of diamond property:
  - When a step Y is enabled in a marking M, the binding elements of Y can occur one by one in any order.
  - The order has no influence on the total effect.

Questions about CPN semantics

A. Can a transition change the marking of places that are neither input nor output places?

B. Can a transition occur concurrently with itself?

C. Can a binding element occur concurrently with itself?

D. Can two transitions that "reads" the value of the same token occur concurrently?

E. Can a marking be reachable from itself?

F. Can we have more than one initial marking?

G. Can we have an infinite number of reachable markings?

Find those where the answer is YES?

Questions