PARAMETER ESTIMATION – 3

Stochastic processes
Discrete time stochastic dynamic models

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Lectures and tutorials

- Basic notions, Elements of random variables and mathematical statistics
- The properties of the estimates, Linear regression
- Stochastic processes, Discrete time stochastic dynamic models
- Least squares (LS) estimation by minimizing the prediction error, The properties of the LS estimation
- Special methods for LS estimation of dynamic model parameters: Instrumental variable (IV) method, Parameter estimation of dynamic nonlinear models
- Practical implementation of parameter estimation: Data checking and preparation, Evaluation of the results of parameter estimation

Lecture overview

1. Discrete time stochastic processes
   - Stochastic processes
   - Mean value and covariance
   - White noise processes
2. Dynamic models of discrete time systems
   - DT-LTI SISO I/O system models
   - DT-LTI stochastic SISO I/O model
3. The principle of parameter estimation – dynamic case
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Overview

1. Discrete time stochastic processes
   - Stochastic processes
   - Mean value and covariance
   - White noise processes
2. Dynamic models of discrete time systems
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Stochastic processes – 1

Stochastic processes are used for describing random disturbances in systems and control theory.

Important

Stochastic process

family (indexed sequence) of random variables $x(.,.)$ *where*

$$x : T \times \Omega \to \mathbb{R}^p$$

The set T *is called time.*

- continuous time *process:* $T \subseteq \mathbb{R}$
- discrete time process*:* $T \subseteq \mathbb{N}$
  discrete time variable $k \sim t_k$

Stochastic processes – 2

Given a discrete time stochastic process

$$x : T \times \Omega \to \mathbb{R}^p$$

- Realization
  the (deterministic) function $x(., \omega_0)$ with ω_0 being fixed
- Fixed-time value
  $x(k_0, .)$ with k_0 is being fixed is a random variable
- Notation
  $x(k, .) = x(k)$ for the random variable generated from the stochastic process x by fixing the time at k

Distribution functions of a stochastic process

A stochastic process can be specified by describing all of its finite dimensional distribution functions

Definition

A finite dimensional distribution function of a stochastic process is defined by the formulae

$$F(\zeta_1, ..., \zeta_n; k_1, ..., k_n) = P\{x(k_1) \leq \zeta_1, ..., x(k_n) \leq \zeta_n\}$$

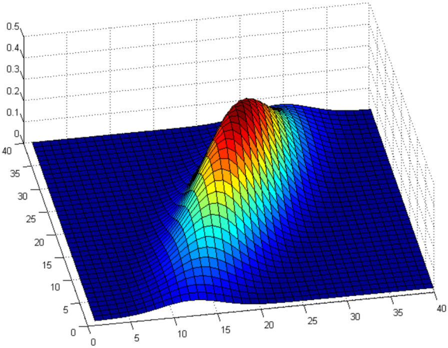
Gaussian or normal process: all finite dimensional distribution functions of the process are Gaussian.

NOTES

Recall: probability distribution function of vector-valued random variables

Two dimensional Gaussian distribution
Probability density function:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{(x_1-m_1)^2}{\sigma_1^2} - 2r\frac{(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2} + \frac{(x_2-m_2)^2}{\sigma_2^2}\right)}$$



Recall: Mean value, covariance

The mean value and variance of the random variable ξ *with its p.d.f.* f_ξ *are*

$$E\{\xi\} = \int x f_\xi(x) dx \quad , \quad \sigma^2\{\xi\} = \int (x - E\{\xi\})^2 f_\xi(x) dx$$

The covariance of two scalar-valued random variables ξ *and* θ *is*

$$COV\{\xi, \theta\} = E\{(\xi - E\{\xi\})(\theta - E\{\theta\})\}$$

Important

The covariance of a scalar-valued random variables ξ *with itself is its variance, i.e.* $COV\{\xi, \xi\} = \sigma^2\{\xi\}$

Mean value function, (auto)covariance function

Definition (mean value function)

The mean-value function of the stochastic process $\{x(k)\}_{k=0}^{\infty}$ is as follows

$$m_x(k) = Ex(k) = \int_{-\infty}^{\infty} \zeta dF(\zeta, k) \quad , \; k = 0, ..., K, ...$$

Important

Note that $m_x(k)$ is an ordinary (deterministic) function of time k.

Definition ((auto)covariance function)

The (auto)covariance function of the stochastic process $\{x(k)\}_{k=0}^{\infty}$ is defined as

$$r_{xx}(\ell, k) = cov \; [x(\ell), x(k)] = E\{ \; [x(\ell) - m(\ell)][x(k) - m(k)]^T \; \}$$

The covariance function is a deterministic two-variate function.

Cross-covariance function

Cross-covariance characterizes the inter-dependence of two discrete time stochastic processes.

Definition (cross-covariance function)

The cross-covariance function of the stochastic processes $\{x(k)\}_{k=0}^{\infty}$ $\{x(k)\}_{k=0}^{\infty}$ and $\{y(k)\}_{k=0}^{\infty}$ is defined as

$$r_{xy}(\ell, k) = cov\ [x(\ell), y(k)] = E\{\ [x(\ell) - m_x(\ell)][y(k) - m_y(k)]^T\ \}$$

The cross-covariance function is a deterministic two-variate function.

White noise processes

Definition (discrete time white noise, e)

A stochastic process $e = \{e(k)\}_{k=-\infty}^{\infty}$ is a discrete time white noise process if it is a sequence of identically distributed, independent random variables.

Important

Properties

- *stationary process (usually $m(k) = 0$ is assumed)*
- *the covariance function in real-valued case is*

$$r_{ee}(\ell) = cov\ [e(k), e(k-\ell)] = \begin{cases} \sigma^2 & \ell = 0 \\ 0 & \ell = \pm 1, \pm 2, \ldots \end{cases}$$

- *A white noise process is* **not** *necessarily a Gaussian process.*

MA processes

Important (unit time delay operator)

Given a signal (time-dependent sequence) $\{x(k),\ k = ..., -1, 0, 1, ...\}$.
The time delay operator q^{-1} *acts as* $q^{-1}x(k) = x(k-1)$.

Definition (moving average process (MA process))

Let $e = \{\ e(k)\ ,\ k = ..., -1, 0, 1, 2, ...\}$ be a white noise process with variance σ^2. Then the related process $y = \{y(t)\}_{k=-\infty}^{\infty}$ which fulfils

$$y(k) = e(k) + b_1 e(k-1) + ... + b_n e(k-n) = B^*(q^{-1})e(k)$$

is termed a MA process.

Mean value and auto-covariance function of a MA process

$$m_y(k) = 0,\ \ r_{yy}(0) = \sigma^2(1 + b_1^2 + ... + b_n^2),$$
$$r_{yy}(1) = \sigma^2(b_1 + b_1 b_2 + ... + b_{n-1} b_n)$$

AR and ARMAX processes

Definition (autoregressive process (AR process))

With the white noise process $e = \{e(t)\}_{k=-\infty}^{\infty}$ an AR process is defined as follows

$$y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = A^*(q^{-1}) y(k) = e(k)$$

Definition (ARMAX process)

An autoregressive-moving average process with an exogeneous signal (ARMAX process) is a linear combination an AR and MA process extended with an exogeneous signal $u = \{u(k)\}_{k=-\infty}^{\infty}$:

$$A^*(q^{-1}) y(k) = B^*(q^{-1}) u(k) + C^*(q^{-1}) e(k)$$

with $A^*(q^{-1}) = 1 + a_1 q^{-1} + a_n q^{-n}$, $B^*(q^{-1}) = b_0 + b_1 q^{-1} + b_m q^{-m}$, $C^*(q^{-1}) = 1 + c_1 q^{-1} + c_n q^{-n}$ and $m < n$.

NOTES

The so-called "General decomposition theorem" in the theory of stochastic processes shows the importance of ARMA processes.

Important (General decomposition theorem)

Any stationary stochastic process $\eta = \{\eta(k)\}_{k=-\infty}^{\infty}$ *with finite variance enables to construct an ARMA model, i.e. there exists a (non-necessarily Gaussian) white noise process* $e = \{e(k)\}_{k=-\infty}^{\infty}$*, and polynomials* $A^*(q^{-1})$ *and* $B^*(q^{-1})$ *such that*

$$A^*(q^{-1})\eta(k) = B^*(q^{-1})e(k)$$

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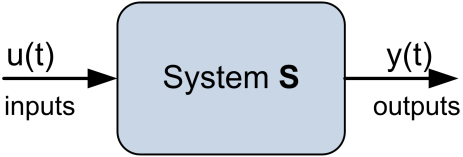
Systems

System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

- inputs (u) and outputs (y)



Basic system properties

- Linearity

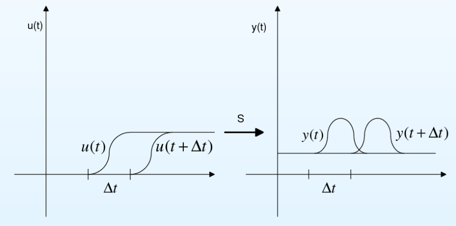
$$\mathbf{S}[c_1 u_1 + c_2 u_2] = c_1 y_1 + c_2 y_2$$

with $c_1, c_2 \in \mathbb{R}, \quad u_1, u_2 \in \mathcal{U}, \quad y_1, y_2 \in \mathcal{Y}$ and $\mathbf{S}[u_1] = y_1, \quad \mathbf{S}[u_2] = y_2$
Linearity check: use the definition

- Time-invariance

$$\mathbf{T}_\tau \circ \mathbf{S} = \mathbf{S} \circ \mathbf{T}_\tau$$

where $\mathbf{T}_\tau$ is the time-shift operator: $\mathbf{T}_\tau(u(t)) = u(t+\tau), \quad \forall t$
Time invariance check: **constant parameters**



Discrete time LTI SISO I/O system models

Discrete difference equation models: for SISO (single-input single-output) systems

- Backward difference form

$$y(k) + a_1 y(k-1) + \dots + a_n y(k-n) = b_d u(k-d) + \dots + b_m u(k-m)$$

where $d = n - m > 0$ is the *pole excess (time delay)*.

- *Compact form*

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d)$$

where $A^*(q^{-1}) = 1 + a_1 q^{-1} + \dots + a_n q^{-n}$ and $B^*(q^{-1}) = b_0 + b_1 q^{-1} + \dots + b_m q^{-m}$ are polynomials of the time delay operator q^{-1}.

Discrete time LTI stochastic SISO I/O model

Important (discrete time stochastic LTI input-output model)

The general form of the input-output model of discrete time stochastic LTI SISO systems is the following canonical ARMAX process:

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k) + C^*(q^{-1})e(k) \tag{1}$$

with the polynomials

$$A^*(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-n},\ C^*(q^{-1}) = c_0 + c_1 q^{-1} + \ldots + c_n q^{-n}$$
$$B^*(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_m q^{-m}$$

where $C^*(q^{-1})$ is assumed to be a stable polynomial.

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ARX models

Important (simplest discrete time stochastic LTI input-output model)

Assuming only independent measurement noise, *the model is an ARX model in the form*

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k) + e(k) \tag{2}$$

where $\{e(k)\}_{k=-\infty}^{\infty}$ *is a white noise process.*

Important (predictive form of ARX models)

The predictive form of the ARX model is

$$y(k) = -a_1 y(k-1) - .. - a_n y(k-n) + b_0 u(k) + ... +$$
$$+ b_m u(k-m) + e(k) = p^T \varphi(k) + e(k)$$

This model is **linear in parameters** $p = [-a_1 \; ... \; -a_n \mid b_0 \; ...b_m]^T$ *if one measures the data*

$$\varphi(k) = [y(k-1) \; ... \; y(k-n) \mid u(k) \; ... \; u(k-m)]^T$$

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Tutorial problems
Stochastic processes

- A. Moving average processes
- B. Two stochastic processes

Tutorial problems – A

Example (Simple MA process – 1)

Given a scalar-valued white noise stochastic process $\{e(k)\}_{-\infty}^{\infty}$ with variance σ^2. Let us construct from it a stochastic process by the equation

$$y(k) = e(k) + 0.5e(k-1) + 0.6e(k-2)$$

- *What kind of process is the stochastic process $\{y(k)\}_{-\infty}^{\infty}$?*
  A moving average (MA) process
- *Compute the mean value function $m_y(k)$ and the (auto)covariance function $r_{yy}(k)$ of the stochastic process $\{y(k)\}_{-\infty}^{\infty}$.*
  $m_y(k) \equiv 0$ *for* $k + 0, 1, ...$
  $r_{yy}(0) = \sigma^2(1 + 0.5^2 + 0.6^2)$, $r_{yy}(\pm 1) = \sigma^2(0.5 + 0.5 \cdot 0.6)$
  $r_{yy}(\pm 2) = \sigma^2 \cdot 0.6$, $r_{yy}(\pm \ell) = 0$, $\ell > 2$

Tutorial problems – A

Example (Simple MA process – 2)

Consider the following stochastic process:

$$w(k) = z(k) + 0.1z(k-1) + 0.8z(k-3)$$

where z is a sequence of independent scalar valued random variables with the same distribution, $E(z(k)) = 0$, and $D(z(k)) = \sigma$, for every k.

- *What kind of process is the stochastic process $\{z(k)\}_{-\infty}^{\infty}$?*
  A white noise process
- *What kind of process is the stochastic process $\{w(k)\}_{-\infty}^{\infty}$?*
  A moving average (MA) process
- *Compute the (auto)covariance function $r_{ww}(k)$ for $k = 1, 3, -2$.*
  $m_w(k) \equiv 0$ *for* $k + 0, 1, \ldots$
  $r_{ww}(1) = \sigma^2 \cdot 0.1$, $r_{ww}(3) = \sigma^2 \cdot 0.8$
  $r_{ww}(-2) = \sigma^2 \cdot 0.1 \cdot 0.8$

Tutorial problems – B

Example (Cross-covariance)

Consider the following two moving-average (MA) processes:

$$\begin{aligned} z(k) &= e(k) + 0.6e(k-1) + 0.1e(k-2) \\ y(k) &= e(k) + 0.3e(k-1) + 0.8e(k-2) \end{aligned}$$

where $\{e(k)\}_{-\infty}^{\infty}$ *is a discrete time white noise process with variance* $D^2(e(k)) = \sigma^2$

Compute the cross-covariance function $r_{zy}(k)\ \forall k$

$m_z(k) \equiv 0 \quad , \quad m_z(k) \equiv 0 \quad , \quad r_{zy}(k) \neq r_{zy}(-k) \qquad$ *!!!*

- $r_{zy}(0) = \sigma^2(1 + 0.6 \cdot 0.3 + 0.1 \cdot 0.8)$
- $r_{zy}(1) = \sigma^2(1 \cdot 0.6 + 0.1 \cdot 0.3) \quad , \quad r_{zy}(-1) = \sigma^2(1 \cdot 0.3 + 0.6 \cdot 0.8)$
- $r_{zy}(2) = \sigma^2 \cdot 1 \cdot 0.1 \quad , \quad r_{zy}(-2) = \sigma^2 \cdot 1 \cdot 0.8$
- $r_{zy}(k) = r_{zy}(-k) = 0 \quad , \quad |k| > 2$

HOMEWORK

Given a scalar-valued white noise stochastic process $\{e(k)\}_{-\infty}^{\infty}$ with variance σ^2. Let us construct from it a stochastic process by the equation

$$y(k) = e(k) - 0.2e(k-1)$$

- What kind of process is the stochastic process $\{y(k)\}_{-\infty}^{\infty}$?
- Compute the mean value function $m_y(k)$ and the (auto)covariance function $r_{yy}(k)$ of the stochastic process $\{y(k)\}_{-\infty}^{\infty}$ for the values $k = 0, \pm 1, \pm 2, \pm 3, ...$!
- Compute the cross-covariance function $r_{ye}(k)$ for the values $k = 0, \pm 1, \pm 2, \pm 3, ...$!

The solution should be submitted electronically
by 12:00 on the 21th October 2020
to the e-mail address hangos.katalin@virt.uni-pannon.hu