Számítógépvezérelt szabályozások elmélete

Irányíthatóság és megfigyelhetőség

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Overview

CT-LTI state-space models

- General form - revisited

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)$$
$$y(t) = Cx(t)$$

  with
  - signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
  - system parameters: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{p \times n}$ $(D = 0)$
- Dynamic system properties:
  - controllability
  - observability
  - stability

Transformation of states

Two different state space models with the same input-output behavior

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad \dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t)$$
$$y(t) = Cx(t) + Du(t) \quad , \quad y(t) = \overline{C}\overline{x}(t) + \overline{D}u(t)$$

which are related by the transformation

$$T \in \mathbb{R}^{n\times n} \;\; , \;\; det\ T \neq 0 \;\; , \;\; \overline{x} = Tx \quad \Rightarrow \quad x = T^{-1}\overline{x}$$

$$dim\ \mathcal{X} = dim\ \overline{\mathcal{X}} = n$$

$$T^{-1}\dot{\overline{x}} = AT^{-1}\overline{x} + Bu$$

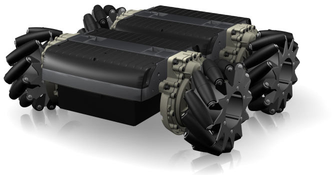
$$\dot{\overline{x}} = TAT^{-1}\overline{x} + TBu \quad , \quad y = CT^{-1}\overline{x} + Du$$

$$\overline{A} = TAT^{-1} \quad , \quad \overline{B} = TB \quad , \quad \overline{C} = CT^{-1} \quad , \quad \overline{D} = D$$

Overview

Controllability of CT-LTI systems

- Problem statement
  - *Given*:
    - a state-space model with parameters (A, B, C)
    - an initial state $x(t_1)$ and a final state $x(t_2) \neq x(t_1)$
  - *Compute*:
    an input signal $u(t)$ which moves the system from $x(t_1)$ to $x(t_2)$ in finite time



Controllability of CT-LTI systems

Theorem (Controllability)

Given (A, B, C) for

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

This SSR with state space $\mathcal{X}$ is state controllable ***iff*** *the* controllability matrix $\mathcal{C}_n$ *is of* ***full rank***

$$\mathcal{C}_n = \begin{bmatrix} B & AB & A^2B & . & . & A^{n-1}B \end{bmatrix}$$

Kalman rank condition: If $\dim\mathcal{X} = n$ then rank $\mathcal{C}_n = n$.

- Necessary and sufficient condition

Controllability of CT-LTI systems

- *Proof:* (constructive)
  - Apply the Dirac-delta function as input to the system, i.e. $u(t) = \delta(t)$ with $C = I$

$$x(t) = h(t) = e^{At}B \quad , \quad y(t) = x(t) \quad , \quad x(0_-) = h(0_-) = B$$

  - Then with $\dot{h}(t) = Ah(t)$

$$\begin{aligned}
\mathbf{S}[u(t) = \delta(t)] &= h(t) \\
\mathbf{S}[u(t) = \dot{\delta}(t)] &= \dot{h}(t) = Ah(t) \\
\mathbf{S}[u(t) = \ddot{\delta}(t)] &= \ddot{h}(t) = A^2 h(t) \\
&\vdots
\end{aligned}$$

  - Assume the **input**: $u(t) = g_1\delta(t) + g_2\dot{\delta}(t) + ... + g_n\delta^{(n-1)}(t)$
  - The superposition principle gives:

$$x(0_+) = x(0_-) + g_1 h(0_-) + g_2\dot{h}(0_-) + ... + g_n h^{(n-1)}(0_-)$$
$$x(0_+) = x(0_-) + g_1 B + g_2 AB + ... + g_n A^{n-1}B$$

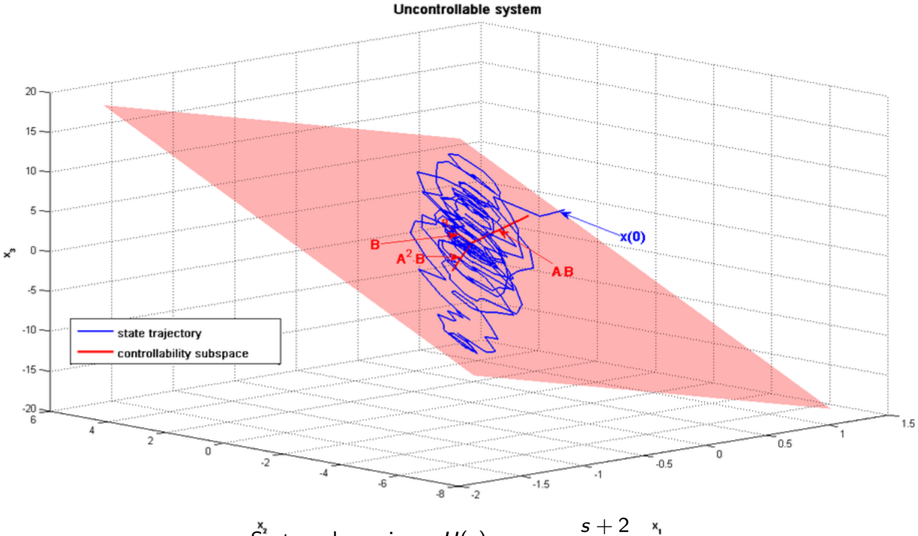
Controllability of CT-LTI systems

- Assuming $x(0_-) = 0$ we get

$$x(0_+) = \left[\begin{array}{ccccccc} B & AB & A^2B & . & . & . & A^{n-1}B \end{array}\right] \left[\begin{array}{c} g_1 \\ g_2 \\ . \\ . \\ . \\ g_n \end{array}\right]$$

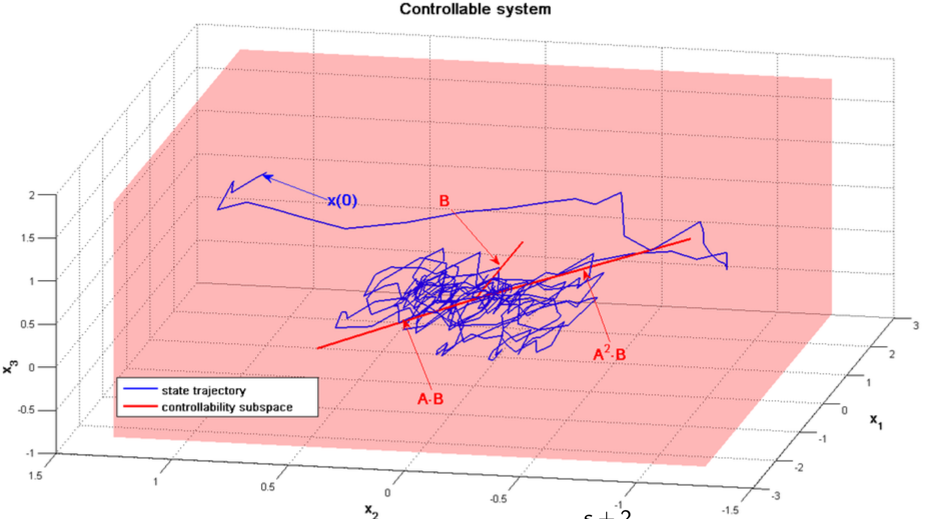
- $x(0_+)$ is an arbitrary desired final state vector then we **can find a unique** $[g_1 ... g_n]^T$ **(for** $u(t)$**) iff** *rank* $\mathcal{C}_{n-1}(A, B) = n$.
- Controllability subspace: subspace spanned by the columns of $\mathcal{C}$
- Controllability is realization dependent since $\mathcal{C} = \mathcal{C}(A, B)$

Uncontrollable state space



System dynamics: $H(s) = \dfrac{s+2}{s^3+4s^2+6s+4}$

Controllable state space

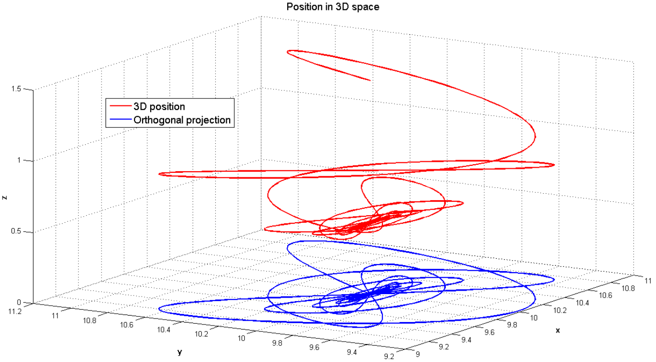


System dynamics: $H(s) = \dfrac{s+2}{s^3+4s^2+6s+4}$

Overview

Observability of CT-LTI systems

- Problem statement
  - *Given*:
    - a state-space model with parameters (A, B, C)
    - a measurement record of $u(t)$ and $y(t)$ as over a finite time interval
  - *Compute*:
    - The state signal $x(t)$ over the finite time interval
    - It is enough to compute $x(t_0) = x_0$



Observability of CT-LTI systems

Theorem (Observability)

Given (A, B, C). *This SSR with state space* $\mathcal{X}$ *is* state observable ***iff*** *the* observability matrix $\mathcal{O}_n$ *is of full rank*

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

Kalman rank condition: If $\dim\mathcal{X} = n$ *then rank* $\mathcal{O}_n = n$.

- A necessary and sufficient condition

Observability of CT-LTI systems

- *Proof:* (constructive)
  - Output and its derivatives can be expressed as

$$
\begin{aligned}
y &= Cx \\
\dot{y} &= C\dot{x} = CAx + CBu \\
\ddot{y} &= C\ddot{x} = CA(Ax + Bu) + CB\dot{u} = CA^2x + CABu + CB\dot{u} \\
&. \\
&. \\
y^{(n-1)} &= Cx^{(n-1)} = CA^{n-1}x + CA^{n-2}Bu + \ldots + CABu^{(n-3)} + CBu^{(n-2)}
\end{aligned}
$$

  - Matrix form

$$
\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ . \\ . \\ . \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ CB & 0 & . & . & . & 0 \\ CAB & CB & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ CA^{n-2}B & CA^{n-3}B & . & . & CB & 0 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \\ . \\ . \\ . \\ u^{(n-1)} \end{bmatrix}
$$

Observability of CT-LTI systems

- Compact form

$$\dot{\mathcal{Y}}(t) = \mathcal{O}_n x(t) + \mathcal{T}\dot{\mathcal{U}}(t)$$

- Zero initial state conditions

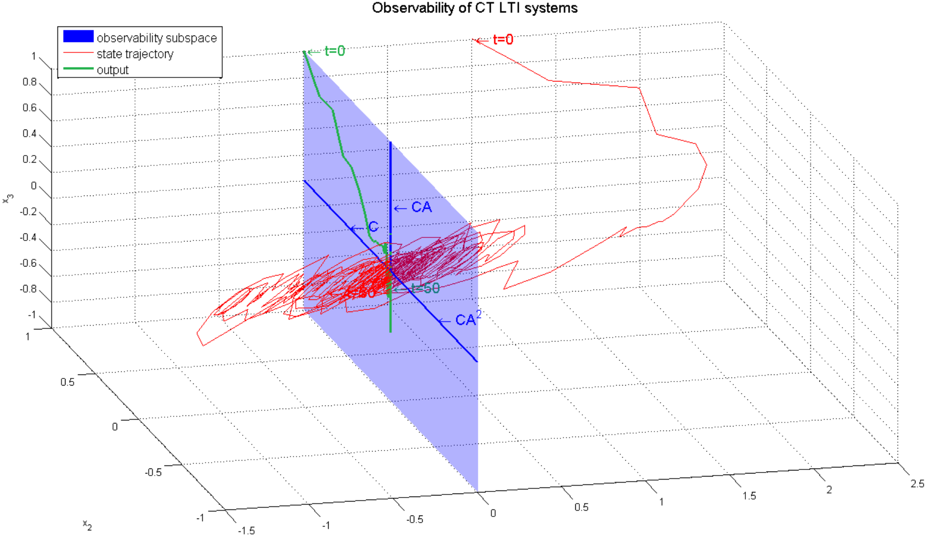
$$\dot{\mathcal{U}} = 0 \quad for \quad t = 0_-$$

- Then

$$\dot{\mathcal{Y}}(0_-) = \mathcal{O}_n(A, C)x(0_-)$$

- $x(0_-)$ can be uniquely determined iff $rank\ \mathcal{O}_n(A, C) = n$.
- Observability subspace: subspace spanned by the rows of $\mathcal{O}$
- Observability is realization dependent since $\mathcal{O} = \mathcal{O}(A, C)$

Unobservable state space



Overview

Controllable canonical form (controller form)

- Applying Laplace transform on the differential equation

$$Y(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} U(s) = \frac{b(s)}{a(s)} U(s) = H(s)U(s)$$

- Let the states be

$$\begin{aligned}
X_1(s) &= \frac{s^{n-1}}{a(s)} U(s) & & \\
X_2(s) &= \frac{s^{n-2}}{a(s)} U(s) & \dot{x}_2(t) &= x_1(t) \\
&\vdots & &\vdots \\
X_n(s) &= \frac{1}{a(s)} U(s) & \dot{x}_n(t) &= x_{n-1}(t)
\end{aligned}$$

- Using the above notation

$$\begin{aligned}
sX_1(s) &= -a_1 X_1(s) - \cdots - a_n X_n(s) + U(s) \\
\dot{x}_1(t) &= -a_1 x_1(t) - \cdots - a_n x_n(t) + u(t) \\
Y(s) &= b_1 X_1(s) + \cdots - b_n X_n(s) \\
y(t) &= b_1 x_1(t) + \cdots + b_n x_n(t)
\end{aligned}$$

Controllable canonical form (controller form)

- Controllability canonical form of the state space model

$$\dot{\underline{x}}(t) = \begin{bmatrix} -a_1 & \dots & -a_{n-1} & -a_n \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} \underline{x}(t)$$

- The change of the i-th state variable depends on the $i-1$-th one, $i > 1$
- The change of x_1 depends on all states and the input
- Always controllable

Observable canonical form

- Applying Laplace transform on the differential equation

$$Y(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} U(s) = \frac{b(s)}{a(s)} U(s) = H(s)U(s)$$

- Let the states be

$$\begin{aligned}
X_1(s) &= Y(s) \\
\\
sX_1(s) &= -a_1 X_1(s) + X_2(s) + b_1 U(s) \\
\dot{x}_1(t) &= -a_1 x_1(t) + x_2(t) + b_1 u(t) \\
\\
sX_2(s) &= -a_2 X_2(s) + X_3(s) + b_2 U(s) \\
\dot{x}_2(t) &= -a_2 x_2(t) + x_3(t) + b_2 u(t) \\
&\vdots \\
sX_n(s) &= -a_n X_n(s) + b_n U(s) \\
\dot{x}_n(t) &= -a_n x_n(t) + b_n u(t)
\end{aligned}$$

Observable canonical form

- Observability canonical form of the state space model

$$\begin{aligned}\dot{\underline{x}}(t) &= \begin{bmatrix} -a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \underline{x}(t)\end{aligned}$$

- Each state variable is fed back to the previous one and the output of the system is x_1.
- Always observable

Diagonal form (or modal form) realization

- State space model in diagonal form

$$\dot{x}(t) = A_D x(t) + B_D u(t)$$
$$y(t) = C_D x(t)$$

with

$$\dot{x} = \begin{bmatrix} \lambda_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & . & . & . & c_n \end{bmatrix} x$$

Controllability in diagonal form realization

- Controllability matrix

$$\mathcal{C}_n = \begin{bmatrix} B & AB & . & . & A^{n-1}B \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & . & . \end{bmatrix} =$$

$$= \begin{bmatrix} b_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & . & . & \lambda^{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \lambda_n & . & . & \lambda_n^{n-1} \end{bmatrix}$$

- The last matrix is a Vandermonde matrix V with determinant

$$det\ V = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$$

- Full rank of the controllability matrix

$$rank\ \mathcal{C}_n = n \quad \Leftrightarrow \quad det\ \mathcal{C}_n = \prod_i b_i \prod_{j<i} (\lambda_i - \lambda_j) \neq 0$$

Controllability and observability in diagonal form realization

Theorem (Controllability)

DSSR is controllable ***iff*** $\lambda_i \neq \lambda_j, (i \neq j)$ ***and*** $b_i \neq 0, \ \forall i$

Theorem (Observability)

DSSR is observable ***iff*** $\lambda_i \neq \lambda_j, (i \neq j)$ ***and*** $c_i \neq 0, \ \forall i$

The transfer function of diagonal form realization

- Transfer function

$$H(s) = C(sI - A)^{-1}B = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)}$$

where I is a unit matrix.

- If either $c_j = 0$ or $b_k = 0$ then the transfer function can be described by smaller number of partial fractions than the original:

$$H(s) = \sum_{i=1}^{\overline{n}} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)} \quad , \quad \overline{n} < n$$