Számítógéppel irányított rendszerek elmélete

Signals and Systems

Hangos Katalin

Villamosmérnöki és Információs Rendszerek Tanszék

Folyamatirányítási Kutató Csoport

MTA Számítástechnikai és Automatizálási Kutató Intézete

Budapest

e-mail: hangos@scl.sztaki.hu

SIGNALS

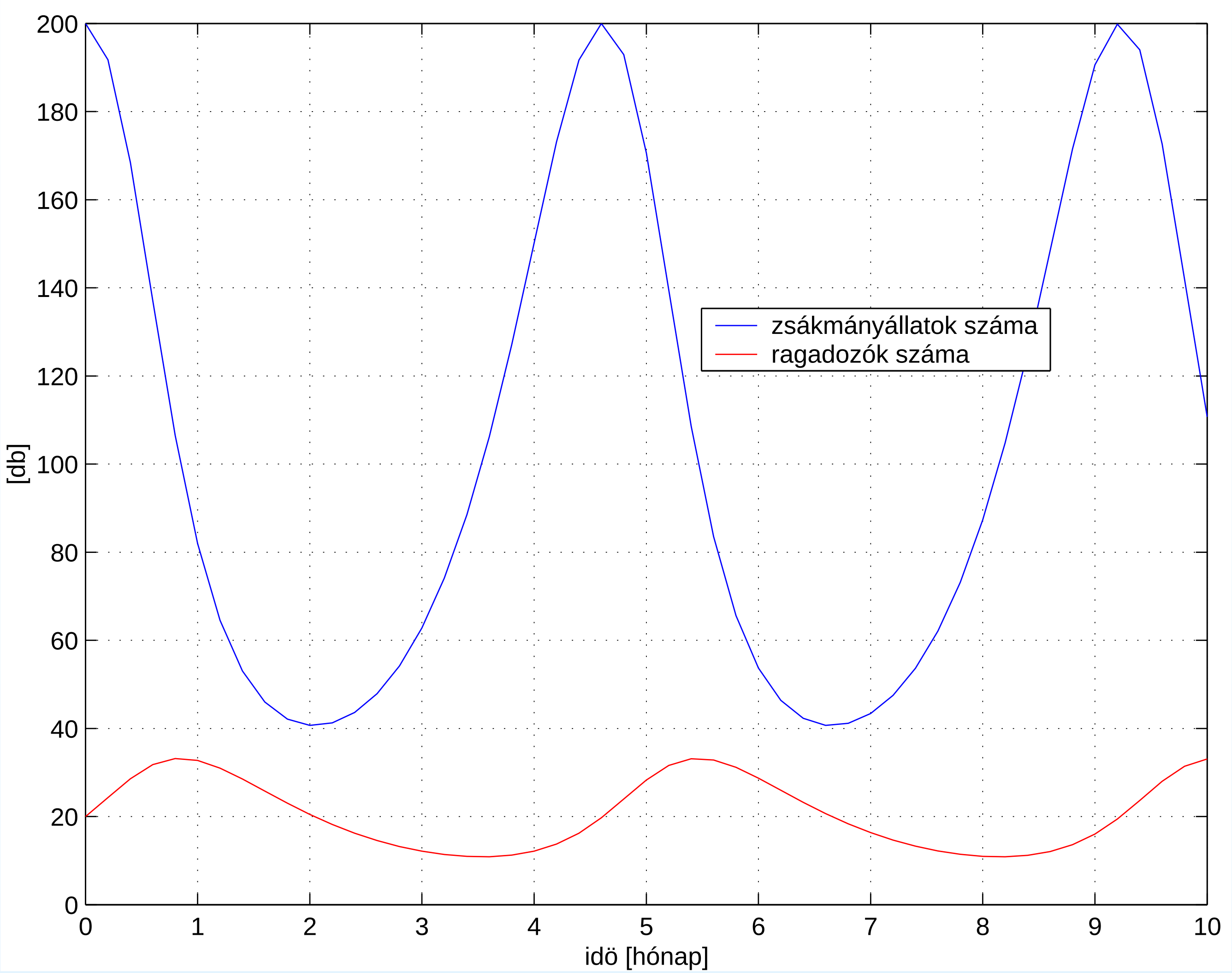
Signals – 1

Signal: time-dependent quantity

Examples

- $x : \mathbb{R}_0^+ \mapsto \mathbb{R}, \quad x(t) = e^{-t}$
- $y : \mathbb{N}_0^+ \mapsto \mathbb{R}, \quad y[n] = e^{-n}$
- $X : \mathbb{C} \mapsto \mathbb{C}, \quad X(s) = \frac{1}{s+1}$

Signals – 2



200
180
160
140
120
100
80
60
40
20
0
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0
1
2
3
4
5
6
7
8
9
10
idö [hónap]
zsákmányállatok száma
ragadozók száma

Signals – 3

- room temperature: $T(x, y, z, t)$
  (x, y, z: spatial coordinates, t: time)
- coloured TV screen: $I : \mathbb{R}^3 \mapsto \mathbb{R}^3$

$$I(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t), \end{bmatrix}$$

Classification of signals

- dimension of the independent variable
- dimension of the signal
- real-valued vs. complex-valued
- continuous time vs. discrete time
- bounded vs. unbounded
- periodic vs. aperiodic
- even vs. odd

Special signals – 1

Dirac-δ or *unit impulse* function

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

where $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$ arbitrary smooth (many times continuously differentiable) function. Consequence:

$$\int_{-\infty}^{\infty} 1 \cdot \delta(t)dt = 1$$

Physical meaning of the unit impulse:

- temperature impulse $\Rightarrow$ energy
- force impulse $\Rightarrow$ momentum
- pressure impulse $\Rightarrow$ mass
- density impulse: mass point

Special signals – 2

Unit step function

$$\eta(t) = \int_{-\infty}^{t} \delta(\tau) d\tau,$$

i.e.

$$\eta(t) = \begin{cases} 0, & \text{ha } t < 0 \\ 1, & \text{ha } t \geq 0 \end{cases}$$

Basic operations on signals – 1

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

- addition:
  $(x+y)(t) = x(t) + y(t), \quad \forall t \in \mathbb{R}_0^+$
- multiplication by scalar:
  $(\alpha x)(t) = \alpha x(t) \quad \forall t \in \mathbb{R}_0^+, \ \alpha \in \mathbb{R}$
- scalar product:
  $\langle x, y \rangle_\nu (t) = \langle x(t), y(t) \rangle_\nu \quad \forall t \in \mathbb{R}_0^+$

Basic operations on signals – 2

- time shift:
  $\mathbf{T}_a x(t) = x(t-a) \quad \forall t \in \mathbb{R}_0^+, a \in \mathbb{R}$
- convolution: $x, y : \mathbb{R}_0^+ \mapsto \mathbb{R}$

$$(x * y)(t) = \int_0^t x(\tau)y(t-\tau)d\tau, \quad \forall t \geq 0$$

Laplace-transformation

Domain:

$$\Lambda = \{\ f \mid f : \mathbb{R}_0^+ \mapsto \mathbb{C}, f \text{ integrable } [0,a]\text{-n } \forall a > 0 \text{ and}$$
$$\exists A_f \geq 0, a_f \in \mathbb{R}, \text{ such that } |f(x)| \leq A_f e^{a_f x}\ \forall x \geq 0\ \}$$

Definition:

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt, \quad f \in \Lambda, \ s \in \mathbb{C}$$

Properties

(1) Linear: $\mathcal{L}\{c_1 y_1 + c_2 y_2\} = c_1\mathcal{L}\{y_1\} + c_2\mathcal{L}\{y_2\}$

(2) $\mathcal{L}\{\frac{dy}{dt}\} = sY(s)$

(3) $\mathcal{L}\{\int_0^t h(t-\tau)u(\tau)d\tau\} = H(s)U(s)$

SYSTEMS

Systems

System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

- inputs (u) and outputs (y)

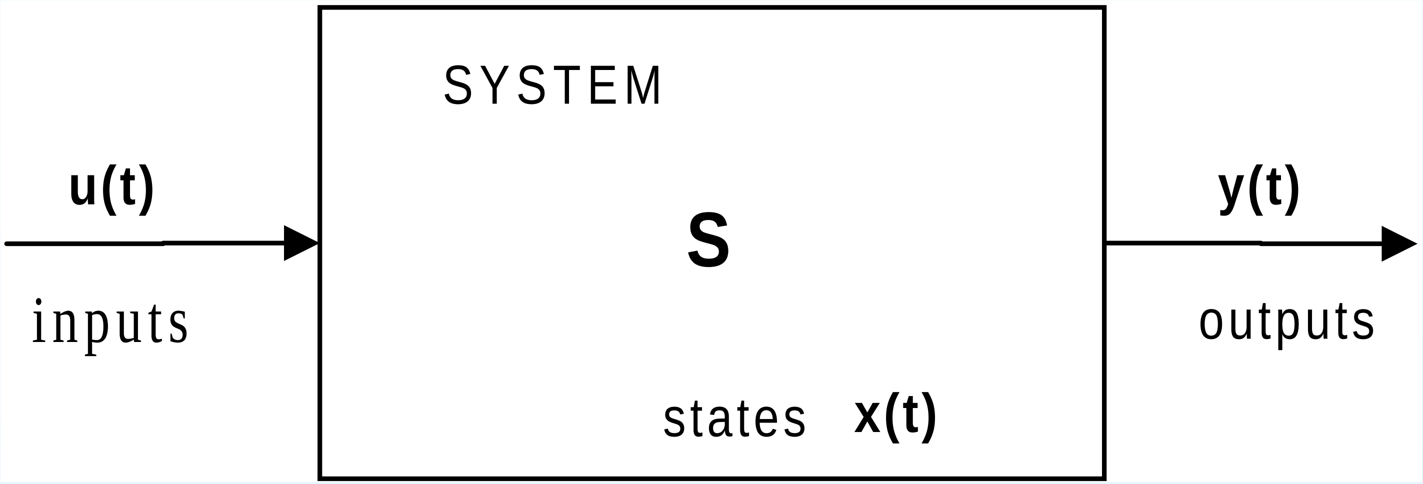


Figure 1: Signal flow diagram of a system

Basic system properties – 1

- *linearity*

$$\mathbf{S}[c_1 u_1 + c_2 u_2] = c_1 y_1 + c_2 y_2$$

with $c_1, c_2 \in \mathcal{R}$, $u_1, u_2 \in \mathcal{U}$, $y_1, y_2 \in \mathcal{Y}$ and
$\mathbf{S}[u_1] = y_1 \quad , \quad \mathbf{S}[u_2] = y_2$

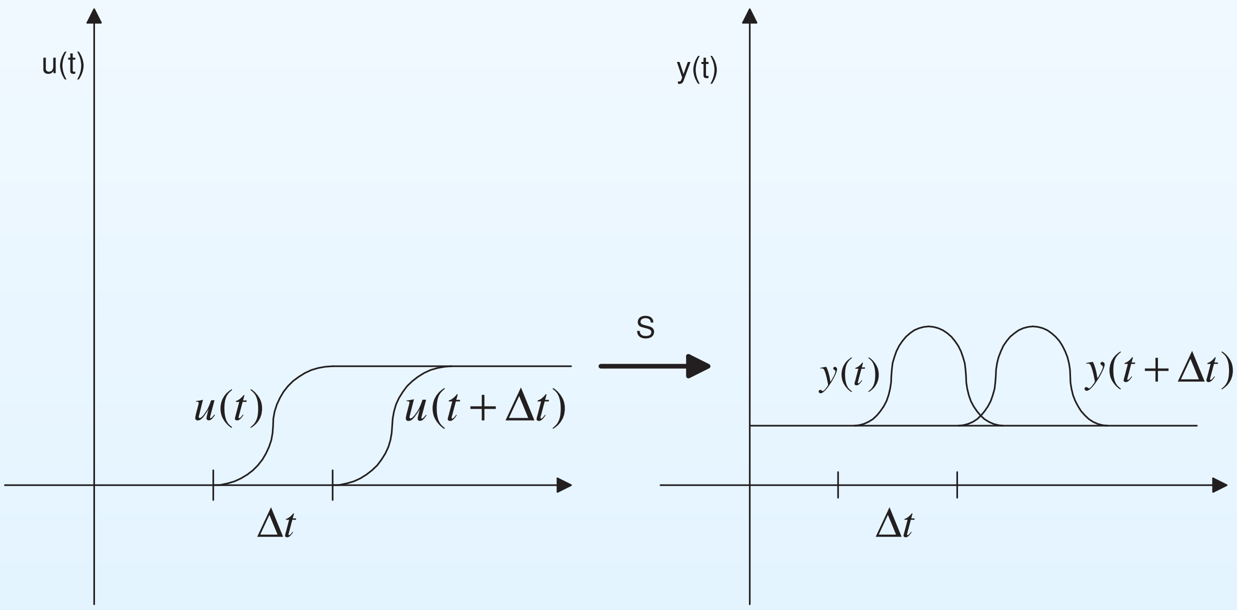
Linearity check: use the definition

Basic system properties – 2

- *time-invariance*

$$\mathbf{T}_\tau \circ \mathbf{S} = \mathbf{S} \circ \mathbf{T}_\tau$$

where $\mathbf{T}_\tau$ is the time-shift operator
Time invariance check: **constant parameters**



Basic system properties – 3

- *continuous and discrete time systems*

  continuous time: ($\mathcal{T} \subseteq \mathcal{R}$)
  discrete time: $\mathcal{T} = \{\cdots, t_0, t_1, t_2, \cdots\}$
- *single-input single-output (SISO) and multiple-input multiple-output (MIMO) systems*
- *causal systems*

CONTINUOUS TIME LINEAR TIME-INVARIANT SYSTEM MODELS

CT-LTI system models

Input-output (I/O) models for SISO systems

- time domain
- operator domain
- frequency domain

State-space models

CT-LTI I/O system models – 1

Time domain

Linear differential equations with constant coefficients

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_0 u + b_1 \frac{du}{dt} + \ldots + b_m \frac{d^m u}{dt^m}$$

with given initial conditions

$$y(0) = y_0 \ , \ \frac{dy}{dt}(0) = y_{10} \ , \ \ldots \ , \ \frac{d^{n-1} y}{dt^{n-1}}(0) = y_{n0}$$

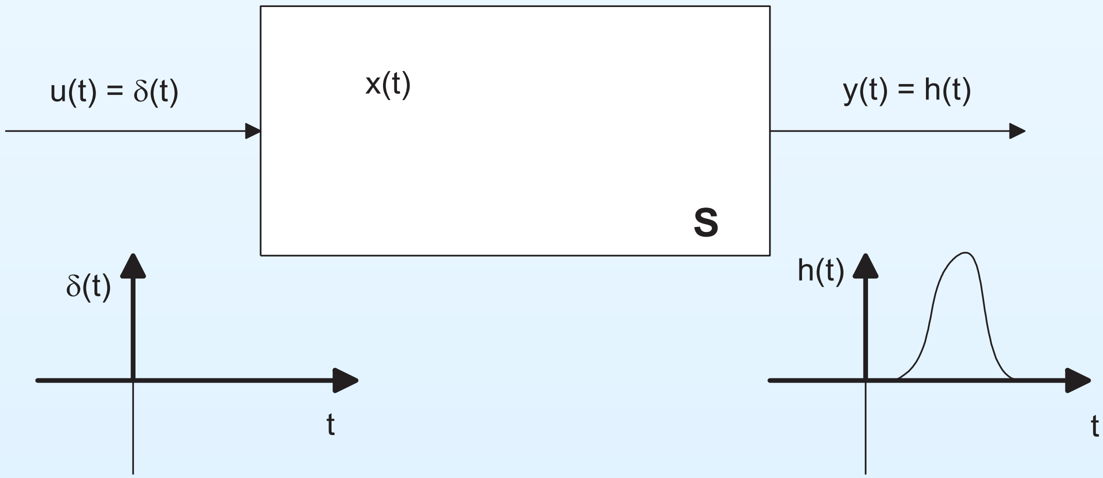
CT-LTI I/O system models – 2

Time domain – *Impulse response function*

is the response of a SISO LTI system to a Dirac-delta input function with zero initial condition.

The output of **S** can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$



CT-LTI I/O system models – 3

Operator domain I/O model for SISO systems

Transfer function

$$Y(s) = H(s)U(s)$$

assuming zero initial conditions with

$Y(s)$	Laplace-transform of the output signal
$U(s)$	Laplace-transform of the input signal
$H(s) = \frac{b(s)}{a(s)}$	*transfer function of the system*
	where $a(s)$ and $b(s)$ are polynomials and
	$degree\ b(s) = m$
	$degree\ a(s) = n$

Strictly proper transfer function: $m < n$

Proper: $m = n$, **inporper**: $m > n$

CT-LTI I/O system models (SISO)

Transfer function – linear diff. equation

$$\mathcal{L}\{a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y\} =$$
$$= \mathcal{L}\{b_0 u + b_1 \frac{du}{dt} + \ldots + b_m \frac{d^m u}{dt^m}\}$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)}$$

Transfer function – Impulse response function

$$H(s) = \mathcal{L}\{h(t)\}$$

CT-LTI state-space models

General form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (\textit{state equation})$$
$$y(t) = Cx(t) + Du(t) \qquad (\textit{output equation})$$

with

- given initial condition $x(t_0) = x(0)$ and $x(t) \in \mathcal{R}^n$,
- $y(t) \in \mathcal{R}^p$, $u(t) \in \mathcal{R}^r$
- system parameters

$$A \in \mathcal{R}^{n\times n}\ ,\ B \in \mathcal{R}^{n\times r}\ ,\ C \in \mathcal{R}^{p\times n}\ ,\ D \in \mathcal{R}^{p\times r}$$

Transformation of states

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad \dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t)$$
$$y(t) = Cx(t) + Du(t) \quad , \quad y(t) = \overline{C}\overline{x}(t) + \overline{D}u(t)$$

which are related by the transformation

$$T \in \mathcal{R}^{n \times n} \quad , \quad det\, T \neq 0 \quad , \quad \overline{x} = Tx \quad \Rightarrow \quad x = T^{-1}\overline{x}$$

$$dim\, \mathcal{X} = dim\, \overline{\mathcal{X}} = n$$

$$T^{-1}\dot{\overline{x}} = AT^{-1}\overline{x} + Bu$$

$$\dot{\overline{x}} = TAT^{-1}\overline{x} + TBu \quad , \quad y = CT^{-1}\overline{x} + Du$$

$$\overline{A} = TAT^{-1} \quad , \quad \overline{B} = TB \quad , \quad \overline{C} = CT^{-1} \quad , \quad \overline{D} = D$$

Computation of the transfer function

Laplace-transformed state-space model

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \qquad (\textit{state equation with} \quad x(0) = 0) \\ Y(s) &= CX(s) + DU(s) \qquad (\textit{output equation}) \end{aligned}$$

$$\begin{aligned} X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= \{C(sI - A)^{-1}B + D\}U(s) \end{aligned}$$

The transfer function $H(s)$ of the SSR (A, B, C, D):

$$H(s) = C(sI - A)^{-1}B + D$$

Solution of the state equation

Apply inverse Laplace-transformation to

$$X(s) = (sI - A)^{-1} BU(s)$$

by using matrix power-series form of $(sI - A)^{-1}$:

$$(sI - A)^{-1} = \frac{1}{s}(I - \frac{A}{s})^{-1} = \frac{1}{s}(I + \frac{A}{s} + \frac{A^2}{s^2} + ...)$$

$$\mathcal{L}^{-1}\{(sI - A)^{-1}\} = I + At + \frac{1}{2!}A^2t^2 + ... = e^{At} \quad , \quad t \geq 0$$

With the equation above we get

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Markov parameters

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Thus the *impulse response function* (with $D = 0$ and $u(t) = \delta(t)$)

$$h(t) = Ce^{At}B = CB + CABt + CA^2B\frac{t^2}{2!} + \ldots$$

Markov parameters

$$CA^iB \quad , \quad i = 0, 1, 2, \ldots$$

are invariant under state transformation.

OUTLOOK: MORE GENERAL SYSTEM MODELS

Generalized CT-LTI state-space models - 1

Linear time-varying (LTV) systems
Generalized CT-LTI case

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (state\ equation)$$
$$y(t) = C(t)x(t) + D(t)u(t) \qquad (output\ equation)$$

- given $x(t_0) = x(0)$ inital conditions, and $x(t) \in \mathcal{R}^n$,
- $y(t) \in \mathcal{R}^p$, $u(t) \in \mathcal{R}^r$
- model parameters: time-dependent matrices

$$A(t) \in \mathcal{R}^{n\times n}\ ,\ B(t) \in \mathcal{R}^{n\times r}\ ,\ C(t) \in \mathcal{R}^{p\times n}\ ,\ D(t) \in \mathcal{R}^{p\times r}$$

Generalized CT-LTI state-space models - 2

Linear parameter varying (LPV) systems
Further generalized CT-LTI case
With a given parameter $\theta(t) \in \mathbb{R}^\ell$

$$\dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) \qquad (state\ equation)$$
$$y(t) = C(\theta(t))x(t) + D(\theta(t))u(t) \qquad (output\ equation)$$

Interpretations

- LTV systems are special cases of LPV models when $\theta(t) = t\ ,\ \ell = 1$,
- linear time-invariant (LTI) systems with time-dependent uncertainty in the parameter $\theta(t)$
- linearized models obtained form nonlinear system models, when the linearization point moves along a state trajectory characterized by the parameter θ

Nonlinear state-space models

Concentrated parameter continuous time case

$$\dot{x}(t) = F(x(t), u(t)) \qquad \text{(state equation)}$$
$$y(t) = G(x(t), u(t)) \qquad \text{(output equation)}$$

with given initial conditions $x(0)$, finite dimensional vectors

$$x(k) \in \mathcal{R}^n \; , \; y(k) \in \mathcal{R}^p \; , \; u(k) \in \mathcal{R}^r$$

and nonlinear functions

$$F : \mathcal{R}^{n+r} \;\mapsto\; \mathcal{R}^n$$

$$G : \mathcal{R}^{n+r} \;\mapsto\; \mathcal{R}^p$$

Special nonlinear systems – 1

Bilinear systems: input-affine systems, where

$$\dot{x}_\ell(t) = \sum_{j=1}^{n} a_{\ell j}^{(0)} x_j(t) \qquad + \textstyle\sum_{j=1}^{m} b_{\ell j}^{(0)} u_j(t) +$$

$$+ \textstyle\sum_{j=1}^{m} \textstyle\sum_{i=1}^{n} b_{ij}^{(\ell)} x_i(t) u_j(t)$$

$$\ell = 1, \dots, n \quad (state\ equation)$$

$$y_k(t) = \sum_{j=1}^{n} c_{kj}^{(0)} x_j(t)$$

$$k = 1, \dots, p \quad (output\ equation)$$

Linear in parameters