Számítógéppel irányított rendszerek elmélete

Joint Controllability and Observability

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BASIC NOTIONS
(from previous lectures)

CT-LTI state-space models

General form - revisited

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)$$
$$y(t) = Cx(t)$$

with

- signals: $x(t) \in \mathcal{R}^n \ , \ y(t) \in \mathcal{R}^p \ , \ u(t) \in \mathcal{R}^r$
- system parameters: $A \in \mathcal{R}^{n \times n} \ , \ B \in \mathcal{R}^{n \times r} \ , \ C \in \mathcal{R}^{p \times n}$ ($D = 0$ by using **centering** the inputs and outputs)

Dynamic system properties:

- observability
- controllability

Observability of CT-LTI systems – 2

A necessary and sufficient condition.

Theorem 1 (O) *Given* (A, B, C)*. This SSR with state space* $\mathcal{X}$ *is* state observable ***iff*** *the* observability matrix $\mathcal{O}_n$ *is of full rank*

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

Kalman rank condition: *If* $dim\mathcal{X} = n$ ***then*** $rank\ \mathcal{O}_n = n$*.*

Controllability of CT-LTI systems – 2

A necessary and sufficient condition.

Theorem 2 (C) *Given* (A, B, C) *for*

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

This SSR with state space $\mathcal{X}$ *is* state controllable ***iff*** *the* controllability matrix $\mathcal{C}_n$ *is of full rank*

$$\mathcal{C}_n = \begin{bmatrix} B & AB & A^2B & . & . & A^{n-1}B \end{bmatrix}$$

Kalman rank condition: *If* $dim\mathcal{X} = n$ *then* $rank\ \mathcal{O}_n = n$*.*

CT-LTI I/O system models – 3

Operator domain I/O model for SISO systems

Transfer function

$$Y(s) = H(s)U(s)$$

assuming zero initial conditions with

$Y(s)$	Laplace-transform of the output signal
$U(s)$	Laplace-transform of the input signal
$H(s) = \frac{b(s)}{a(s)}$	*transfer function of the system*
	where $a(s)$ and $b(s)$ are polynomials and
	$degree\ b(s) = m$
	$degree\ a(s) = n$

$$H(s) = C(sI - A)^{-1}B + D$$

Markov parameters

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Thus the *impulse response function* (with $D = 0$ and $u(t) = \delta(t)$)

$$h(t) = Ce^{At}B = CB + CABt + CA^2B\frac{t^2}{2!} + ...$$

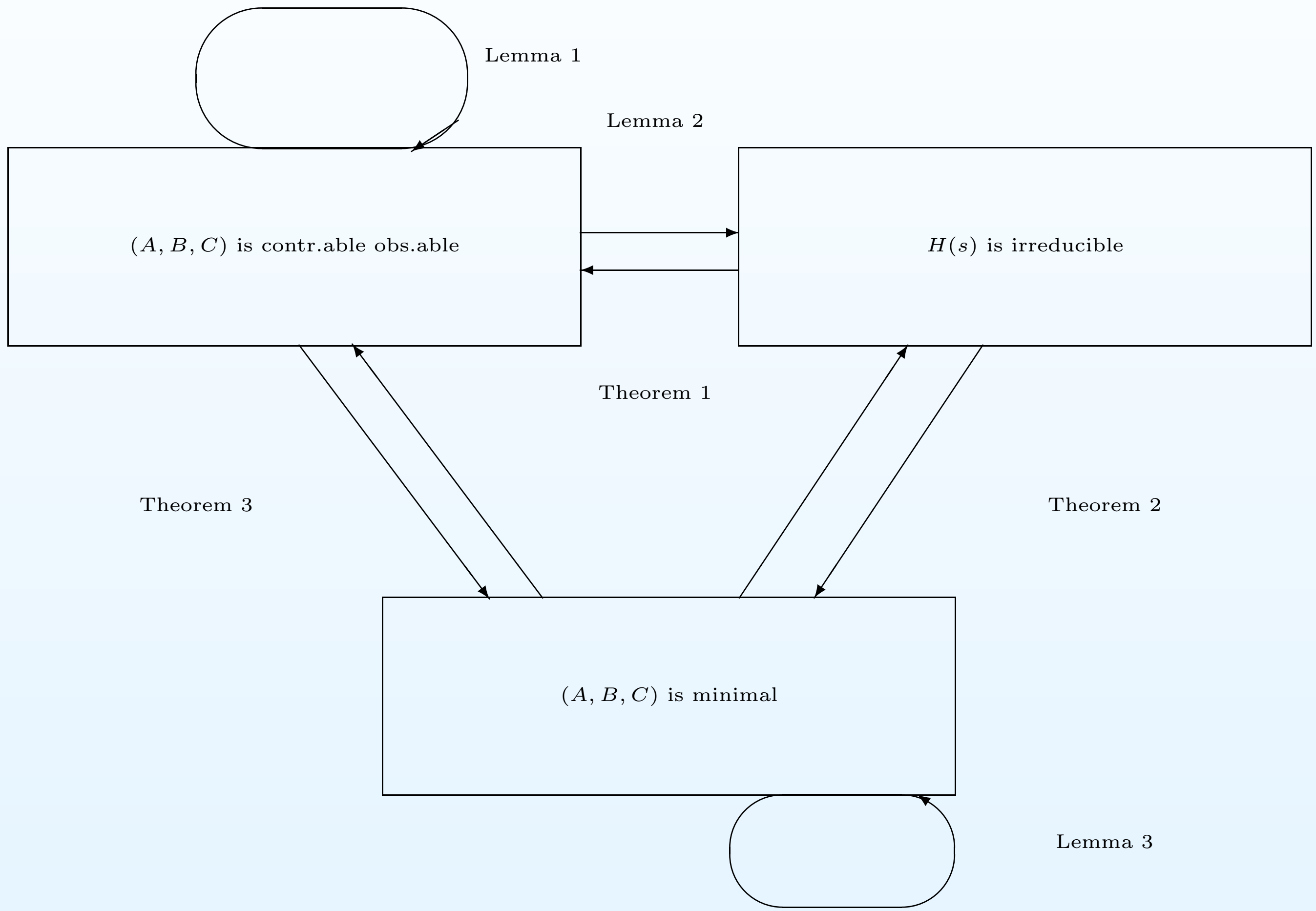
Markov parameters

$$CA^iB \quad , \quad i = 0, 1, 2, ...$$

are invariant under state transformation.

PRELIMINARIES

Overview – 1



equivalent SSR properties

Overview – 2

Consider **SISO** CT-LTI systems with realization (A, B, C)

- Joint controllability and observability is a **system property**
- Equivalent necessary and sufficient conditions
- Minimality of SSRs
- Irreducibility of the transfer function

Hankel matrices

Definition
A Hankel matrix is a block matrix of the following form

$$H[1,n-1] = \begin{bmatrix} CB & CAB & . & . & . & CA^{n-1}B \\ CAB & CA^2B & . & . & . & CA^nB \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ CA^{n-1}B & CA^nB & . & . & . & CA^{2n-2}B \end{bmatrix}$$

It contains *Markov parameters* CA^iB that are invariant under state transformations.

Lemma 1

Lemma 1: *If we have a system with transfer function $H(s) = \frac{b(s)}{a(s)}$ and there is an n-th order realization (A, B, C), which is controllable and observable then all other n-th order realizations are controllable and observable.*

Proof

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} \quad , \quad \mathcal{C}(A,B) = \begin{bmatrix} B & AB & A^2B & . & . & . & A^{n-1}B \end{bmatrix}$$

$$H[1, n-1] = \mathcal{O}(C,A)\mathcal{C}(A,B)$$

REALIZATIONS IN SPECIAL FORM: Controller form realization

Controller form realization

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix}$$

with the coefficients of the polynomials $a(s) = s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n$ and $b(s) = b_1 s^{n-1} + ... + b_{n-1} s + b_n$ that appear in the transfer function $H(s) = \frac{b(s)}{a(s)}$

Definitions

Definition: *relative prime polynomials*

Two polynomials $a(s)$ and $b(s)$ are $coprime$ (or relative primes) iff $a(s) = \prod(s - \alpha_i)$; $b(s) = \prod(s - \beta_j)$ and $\alpha_i \neq \beta_j$ for all i, j.

In other words: the polynomials have no common factors.

Definition: *irreducible transfer function*

A transfer function $H(s) = \frac{b(s)}{a(s)}$ is called to be irreducible if the polynomials $a(s)$ and $b(s)$ are relative primes.

Lemma 2

Lemma 2: *If there exists a controller form realization which is jointly controllable and observable then* $a(s)$ *and* $b(s)$ *are relative primes (*$H(s)$ *is irreducible).*

Proof

1. A controller form realization is controllable and

$$\mathcal{O}_c = \tilde{I}_n b(A_c)$$

$$\tilde{I}_n = \begin{bmatrix} 0 & . & . & 1 \\ 0 & . & 1 & 0 \\ . & . & . & . \\ 1 & 0 & . & 0 \end{bmatrix} \in \mathcal{R}^{n \times n}$$

2. Non-singularity of $b(A_c)$

Proof of Lemma 2 – 1

$$\tilde{I}_n = \begin{bmatrix} e_n & e_{n-1} & . & . & e_1 \end{bmatrix} = \begin{bmatrix} e_n^T \\ e_{n-1}^T \\ . \\ . \\ . \\ e_1^T \end{bmatrix} \quad , \quad e_i = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \end{bmatrix} \leftarrow i.$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix} \quad , \quad e_i^T A_c = \begin{cases} [-a_1 \; - a_2 \; \ldots \; - a_n] & i = 1 \\ e_{i-1}^T & i \geq 2 \end{cases}$$

Proof of Lemma 2 – 2

Computation of the observability matrix $\mathcal{O}_c = \tilde{I}_n b(A_c) \in \mathcal{R}^{n \times n}$

1st row:

$$e_n^T b(A_c) = e_n^T b_1 A_c^{n-1} + ... + e_n^T b_{n-1} A_c + e_n^T b_n I_n$$

n-th term: $\quad [0 \ \ ... \ \ 0 \ \ b_n]$

$(n-1)$-th term: $b_{n-1} e_n^T A_c = b_{n-1} e_{n-1}^T = [0 \ \ ... \ \ b_{n-1} \ \ 0]$

...

$$e_n^T b(A_c) = [b_1 \ \ ... \ \ b_{n-1} \ \ b_n] = C_c$$

2nd row:

$$e_{n-1}^T b(A_c) = e_n^T A_c b(A_c) = e_n^T b(A_c) A_c \ \ \Rightarrow \ \ e_{n-1}^T b(A_c) = C_c A_c$$

and so on ...

Proof of Lemma 2 – 3

$\mathcal{O}_c$ is nonsingular

- iff $b(A_c)$ is nonsingular because matrix $\tilde{I}_n$ is always nonsingular
- $b(A_c)$ is nonsingular iff $det(b(A_c)) \neq 0$
  which depends on the eigenvalues of $b(A_c)$ matrix
- the eigenvalues of the matrix $b(A_c)$ are $b(\lambda_i), \quad i = 1, 2, ..., n$
  λ_i is an eigenvalue of A_c, i.e a root of $a(s) = det(sI - A)$

$$det(b(A_c)) = \prod_{i=1}^{n} b(\lambda_i) \neq 0$$

$\Updownarrow$

$a(s)$ and $b(s)$ have no common roots, i.e. they are relative primes

Minimal realization conditions – 1

Theorem 1: $H(s) = \frac{b(s)}{a(s)}$ *is irreducible iff all* n*-th order realizations are jointly controllable and observable.*

Proof: combine Lemma 1. and 2. •

Definition: *minimal realization*
A realization (A, B, C) of dimension n is minimal if one cannot find another realization of dimension less than n.

Theorem 2: $H(s) = \frac{b(s)}{a(s)}$ *is irreducible iff any of its realization* (A, B, C) *is minimal where* $H(s) = C(sI - A)^{-1}B$

Proof: by contradiction

Minimal realization conditions – 2

Theorem 3.: *A realization (A, B, C) is minimal iff the system is jointly controllable and observable.*

Proof: Combine Theorem 1 and Theorem 2 . •

Lemma 3.: *Any two minimal realizations can be connected by a unique similarity transformation (which is invertible).*

Proof: (Just the idea of it)

$$T = \mathcal{O}^{-1}(C_1, A_1)\mathcal{O}(C_2, A_2) = \mathcal{C}(A_1, B_1)\mathcal{C}^{-1}(A_2, B_2)$$

exists and it is invertible: this is used as a transformation matrix.

General decomposition theorem – 1

(A, B, C) then we can always transform it to another realization $(\overline{A}, \overline{B}, \overline{C})$ with partitioned state vector and matrices

$$\overline{x} = \begin{bmatrix} \overline{x}_{co} & \overline{x}_{c\overline{o}} & \overline{x}_{\overline{c}o} & \overline{x}_{\overline{co}} \end{bmatrix}^T$$

$$\overline{A} = \begin{bmatrix} \overline{A}_{co} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{\overline{c}o} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{\overline{co}} \end{bmatrix} \quad \overline{B} = \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} \overline{C}_{co} & 0 & \overline{C}_{\overline{c}o} & 0 \end{bmatrix}$$

General decomposition theorem – 2

The partitioning defines **subsystems**

- *Controllable and observable subsystem*: $(\overline{A}_{co}, \overline{B}_{co}, \overline{C}_{co})$ **is minimal**, i.e. $\overline{n} \leq n$ and

$$H(s) = \overline{C}_{co}(s\overline{I} - \overline{A}_{co})^{-1}\overline{B}_{co} = C(sI - A)^{-1}B$$

- *Controllable subsystem*

$$\left(\begin{bmatrix} \overline{A}_{co} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} \end{bmatrix} , \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \end{bmatrix} , \begin{bmatrix} \overline{C}_{co} & 0 \end{bmatrix} \right)$$

- *Observable subsystem*

$$\left(\begin{bmatrix} \overline{A}_{co} & \overline{A}_{13} \\ 0 & \overline{A}_{\overline{c}o} \end{bmatrix} , \begin{bmatrix} \overline{B}_{co} \\ 0 \end{bmatrix} , \begin{bmatrix} \overline{C}_{co} & \overline{C}_{\overline{c}o} \end{bmatrix} \right)$$

- *Uncontrollable and unobservable subsystem*

$$([\overline{A}_{\overline{co}}] \ , \ [0] \ , \ [0])$$

OUTLOOK: CONTROLLABILITY OF NONLINEAR SYSTEMS

Fed-batch bioreactor (fermenter)

Controllability of CT-LTI systems

Applying the "brute-force" Dirac-delta input we get

$$x(0_+) = x(0_-) + \begin{bmatrix} B & AB & A^2B & . & . & . & A^{n-1}B \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ . \\ . \\ . \\ g_n \end{bmatrix}$$

If $rank\, \mathcal{C}_{n-1}(A, B) = r$ is not full then we can only move inside a linear sub-space of $\mathbb{R}^n$ of dimension r.

Fed-batch case: state equations

Nonlinear input-affine state-space model

$$\dot{x} = f(x) + g(x)u$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} X \\ S \\ V \end{bmatrix} \quad , \quad u = F$$

$$f(x) = \begin{bmatrix} \mu(x_2)x_1 \\ -\frac{1}{Y}\mu(x_2)x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\mu_{max}x_2x_1}{k_1+x_2+k_2x_2^2} \\ -\frac{\mu_{max}x_2x_1}{(k_1+x_2+k_2x_2^2)Y} \\ 0 \end{bmatrix} \quad , \quad g(x) = \begin{bmatrix} -\frac{x_1}{x_3} \\ \frac{S_f-x_2}{x_3} \\ 1 \end{bmatrix}$$

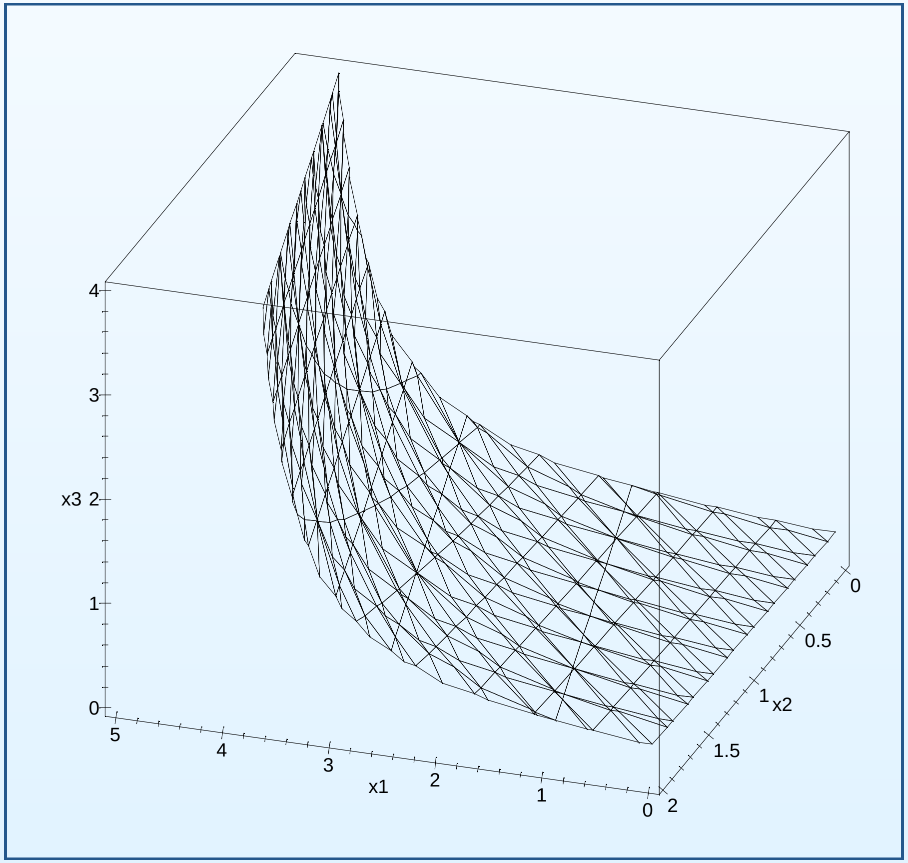
and

$$\mu(x_2) = \frac{\mu_{max}x_2}{k_1 + x_2 + k_2x_2^2}$$

Controllability analysis

$$rank\ \Delta_c = 2\ <\ dim\ x = 3$$

The **reachability hyper-surface** of the fed-batch fermenter for **initial conditions** $x_1(0) = 2\frac{g}{l}$, $x_2(0) = 0.5\frac{g}{l}$, $x_3(0) = 0.5\frac{g}{l}$



Co-ordinate transformation

"Hidden conserved quantity" generating the transformation

$$\lambda(x_1, x_2, x_3) = V(S_f - S) + \frac{1}{Y} V(X_f - X)$$

Transformed "minimal" model ($z_1 = x_1,\ z_2 = x_2,\ z_3 = \lambda(x_1, x_2, x_3)$)

$$\dot{z} = \bar{f}(z) + \bar{g}(z)u$$

where

$$\bar{f}(z) = \begin{bmatrix} \frac{\mu_{max} z_2 z_1}{K_1 + z_2 + K_2 z_2^2} \\ -\frac{\mu_{max} z_2 z_1}{(K_1 + z_2 + K_2 z_2^2)Y} \\ 0 \end{bmatrix}, \quad \bar{g}(z) = \begin{bmatrix} -\frac{z_1}{z_3}(-\frac{1}{Y} z_1 - z_2 + S_f) \\ \frac{S_f - z_2}{z_3}(-\frac{1}{Y} z_1 - z_2 + S_f) \\ 0 \end{bmatrix}$$

Structural properties

- depends on the selection of the input
- does not depend on the source function μ