Számítógépvezérelt szabályozások elmélete

LQ szabályozótervezés

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Feedback

- *state feedback* when the input depends only on states, i.e.

$$u = F(x)$$

- *output feedback* when the input depends only on outputs, i.e.

$$u = F(y)$$

- *static feedback* when the function F is static,
- *linear static feedback* when the function F is a linear static function,
- *full state feedback* when the input signal depends on ***every element in the state vector***.

Closed-loop LTI systems with full state feedback

(A, B, C) of a SISO LTI system **S**

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t) \\
&y(t) = Cx(t) \\
&y(t)\ ,\ u(t) \in \mathcal{R} \quad , \quad x(t) \in \mathcal{R}^n \\
&A \in \mathcal{R}^{n \times n}\ ,\ B \in \mathcal{R}^{n \times 1}\ ,\ C \in \mathcal{R}^{1 \times n} V
\end{aligned}
$$

and linear static full state feedback

$$
\begin{aligned}
&v = u + kx \quad (u = v - kx) \\
&k = \left[\begin{array}{cccccc} k_1 & k_2 & . & . & . & k_n \end{array} \right] \\
&k \in \mathcal{R}^{1 \times n} \qquad (row\ \ vector)
\end{aligned}
$$

LQR: problem statement

Given

- a (MIMO) LTI state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad , \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

- a functional (*control aim*)

$$J(x,u) = \frac{1}{2}\int_0^T [x^T(t)Qx(t) + u^T(t)Ru(t)]dt$$

with $Q^T = Q,\ Q > 0$ and $P^T = P,\ P > 0.$

Compute a control $\{u(t)\ ,\quad t \in [0, T]\}$ that minimizes J subject to the state-space model.

Calculus of variations – 1

Problem statement:
Minimize

$$J(x,u) = \int_0^T F(x,u,t)dt$$

with respect to u subject to $\dot{x} = f(x,u,t)$.
Solution: by using a vector Lagrange multiplier $\lambda(.)$

$$J(x,\dot{x},u) = \int_0^T [F(x,u,t) + \lambda^T(t)(f(x,u,t) - \dot{x})]dt$$

and the Hamiltonian function $H = F + \lambda^T f$.

$$J = \int_0^T [H - \lambda^T \dot{x}]dt$$

Calculus of variations – 2

$\dot{x}$ is eliminated by integrating in part using

$$[\lambda^T x]_0^T = \int_0^T \dot{\lambda}^T x + \int_0^T \lambda^T \dot{x}$$

then $J = \int_0^T [H - \lambda^T \dot{x}] dt$ transforms to

$$J = \int_0^T [H + \dot{\lambda}^T x] dt - [\lambda^T x]_0^T$$

By a **minimizing** u, arbitrary δu in u and δx in x should produce $\delta J = 0$

$$\delta J = -\lambda^T \delta x|_0^T + \int_0^T \left[(\frac{\partial H}{\partial x} + \dot{\lambda}^T)\delta x + \frac{\partial H}{\partial u}\delta u \right] dt$$

Euler-Lagrange equations

$\delta J = 0$ is zero in

$$\delta J = -\lambda^T \delta x|_0^T + \int_0^T \left[(\frac{\partial H}{\partial x} + \dot{\lambda}^T)\delta x + \frac{\partial H}{\partial u}\delta u \right] dt$$

when

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0 \quad , \quad \frac{\partial H}{\partial u} = 0$$

with the Hamiltonian

$$H = F + \lambda^T f$$

LQR Euler-Lagrange equations

Euler-Lagrange equations with the Hamiltonian $H = F + \lambda^T f$:

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0 \quad , \quad \frac{\partial H}{\partial u} = 0$$

Special problem elements:

$$\begin{aligned} f &= Ax + Bu \\ F &= \tfrac{1}{2}(x^T Q x + u^T R u) \\ H &= \tfrac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu) \end{aligned}$$

LQR Euler-Lagrange equations: with $\frac{\partial}{\partial x}(x^T Q x) = 2x^T Q$

$$\dot{\lambda}^T + x^T Q + \lambda^T A = 0 \quad , \quad \lambda^T(T) = 0$$

$$u^T R + \lambda^T B = 0$$

State and co-state dynamics

Rearranged LQR Euler-Lagrange equations

$$\dot{\lambda} + Qx + A^T\lambda = 0 \quad , \quad \lambda(T) = 0$$
$$u = -R^{-1}B^T\lambda$$

State equation:

$$\dot{x} = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$

Joint matrix-vector form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \quad , \quad \begin{matrix} x(0) = x_0 \\ \lambda(T) = 0 \end{matrix}$$

System dynamics + Hammerstein co-state diff. eq.

LQR: Controllable and Observable case

Lemma * When (A, B) is controllable and (C, A) is observable

$$\lambda(t) = K(t)x(t) \quad , \quad K(t) \in \mathcal{R}^{n \times n}$$

The modified state and co-state equations

$$\begin{aligned} \dot{\lambda} + Qx + A^T \lambda = 0 \quad &\Rightarrow \quad \dot{K}x + K\dot{x} = -A^T Kx - Qx \\ u = -R^{-1}B^T \lambda \quad &\Rightarrow \quad u = -R^{-1}B^T Kx \\ \dot{x} = Ax + Bu \quad &\Rightarrow \quad \dot{x} = Ax - BR^{-1}B^T Kx \end{aligned}$$

$$\dot{K}x + K[A - BR^{-1}B^T K]x + A^T Kx + Qx = 0$$

for any $x(t)$. *Matrix Ricatti Differential Equation* for $K(t)$

$$\dot{K} + KA + A^T K - KBR^{-1}B^T K + Q = 0$$

Stationary case

Special case: stationary solution with $T \to \infty$

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt$$

$$\lim_{t \to \infty} K(t) = K \quad \text{i.e.} \quad \dot{K} = 0$$

Control Algebraic Ricatti Equation (CARE)

$$KA + A^T K - KBR^{-1} B^T K + Q = 0$$

Theorem

(Due to R. Kalman) If (C, A) is observable and (A, B) is controllable then CARE has a unique positive definite symmetric solution K.

LQR and its properties

Solution: *linear static full state feedback*

$$u^0(t) = -R^{-1}B^TKx(t) = -Gx(t)$$

where $G = R^{-1}B^TK$. *Closed loop dynamics*

$$\dot{x} = Ax - BR^{-1}B^TKx = (A - BG)x \quad , \quad x(0) = x_0$$

Properties of the closed-loop system

- the closed-loop system is asymptotically stable no matter what the values of A, B, C, R, Q are, i.e.

$$Re\ \lambda_i(A - BG) < 0 \quad , \quad i = 1, 2, ..., n$$

- specific location of the closed-loop poles depend on the choice of Q and R

LQR servo: problem statement

Aim: to follow a time-dependent reference signal $r(t)$

Given : the state equation of an ***extended*** LTI system model

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$
$$\dot{z}(t) = r(t) - y(t) = r(t) - Cx(t)$$

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

In steady-state $\dot{z} = 0$, i.e. $r = y$ or $r = Cx$.

Compute a stabilizing feedback

$$u = -[K_x K_z] \cdot \begin{bmatrix} x \\ z \end{bmatrix}$$

LQR servo: solution

Control gain design: by using pole-placement or LQR design procedure with the extended system parameter matrices

$$A' = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \quad , \quad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Applicability condition: (A', B') should be a controllable pair