Számítógépvezérelt szabályozások elmélete

Megfigyelő tervezés

Magyar Attila

Pannon Egyetem
Műszaki Informatika Kar
Villamosmérnöki és Információs Rendszerek Tanszék

amagyar@almos.vein.hu

2011. február 11.

Observability of CT-LTI systems – 2

A necessary and sufficient condition.

Theorem (O)

Given (A, B, C). *This SSR with state space* $\mathcal{X}$ *is* state observable ***iff*** *the* observability matrix $\mathcal{O}_n$ *is of full rank*

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

Kalman rank condition: If $\dim\mathcal{X} = n$ *then rank* $\mathcal{O}_n = n$.

Observability of CT-LTI systems – 3

Proof: (constructive)

$$
\begin{aligned}
y &= Cx \\
\dot{y} &= C\dot{x} = CAx + CBu \\
\ddot{y} &= C\ddot{x} = CA(Ax + Bu) + CB\dot{u} = CA^2x + CABu + CB\dot{u} \\
&. \\
&. \\
y^{(n-1)} &= Cx^{(n-1)} = CA^{n-1}x + CA^{n-2}Bu + \ldots + CABu^{(n-3)} + CBu^{(n-2)}
\end{aligned}
$$

$$
\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ . \\ . \\ . \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ CB & 0 & . & . & . & 0 \\ CAB & CB & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ CA^{n-2}B & CA^{n-3}B & . & . & CB & 0 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \\ . \\ . \\ . \\ u^{(n-1)} \end{bmatrix}
$$

Observability of CT-LTI systems – 4

The resulting equation in matrix-vector form

$$\dot{\mathcal{Y}}(t) = \mathcal{O}_n x(t) + \mathcal{T}\dot{\mathcal{U}}(t)$$

$x(0_-)$ **can be uniquely determined iff** *rank* $\mathcal{O}_n(A, C) = n$. This is in fact an **observer** but not very fortunate, because the *differentiation of the signals* is needed

Motivation

Consider (A, B, C) of a SISO LTI system **S**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

Apply a **linear static output feedback**

$$u = -ky + v$$

Closed-loop system model

$$\begin{aligned} \dot{x}(t) &= (A - kBC)x(t) + Bv(t) \\ y(t) &= Cx(t) \end{aligned}$$

If the dyadic product BC is rank-deficient (any zero element in B or C) then one cannot stabilize!

Observer form realization - SISO LTI

$$\dot{x}(t) = A_o x(t) + B_o u(t)$$
$$y(t) = C_o x(t)$$

with

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 & . & . & 0 \\ -a_2 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 1 \\ -a_n & 0 & . & . & 0 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 & . & . & . & 0 \end{bmatrix}$$

with the coefficients of the polynomials $a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$ and $b(s) = b_1 s^{n-1} + \ldots + b_{n-1} s + b_n$ that appear in the transfer function $H(s) = \frac{b(s)}{a(s)}$

Observability matrix in observer form

The observability matrix (full rank!)

$$\mathcal{O}_o = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -a_1 & 1 & 0 & \dots & 0 \\ -a_1^2 - a_1 a_2 & -a_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ -* & -* & -* & -* & 1 \end{bmatrix}$$

its inverse

$$\mathcal{O}_o^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{bmatrix}$$

Observer desing for CT-LTI systems

Problem statement *Given*:

- a SISO state-space model with parameters (A, B, C)
- a finite **measurement record** of u and y as signals
- an initial value $\hat{x}_0$

Compute:
An estimate of the state signal x over the finite time interval such that $x(t) \rightarrow \hat{x}(t)$ as $t \rightarrow \infty$

Observer equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

consider the **observer**

$$\frac{\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + L(y - C\hat{x}(t))$$

Introduce the **estimation error** signal: $\check{x} = x - \hat{x}$

$$\frac{\check{x}(t)}{dt} = (A - LC)\check{x}(t)$$

If the matrix $\check{A} = A - LC$ is a stability matrix then $\check{x} \to 0$ when $t \to \infty$ (asymptotic stability). **Task:** find L such that $\check{A} = A - LC$ is a stability matrix

Remainder: pole-placement controller design

Closed-loop SISO system model

$$\dot{x}(t) = (A - Bk)x(t) + Bv(t)$$
$$y(t) = Cx(t)$$

Characteristic polynomials

$$a_c(s) = det\ (sI - A + Bk) \quad := \alpha(s) \quad , \quad a(s) = det\ (sI - A)$$

Task: find k such that $\bar{A} = A - Bk$ has given eigenvalues determined by $a_c(s)$ If **S** is ***controllable*** then

$$k = (\underline{\alpha} - \underline{a})T_{\ell}^{-T}\mathcal{C}^{-1}$$

Duality of state feedback controller and observer

Since the **eigenvalues of a matrix and its transpose are the same**

$$A \leftrightarrow A^T \quad , \quad B \leftrightarrow C^T \quad , \quad k \leftrightarrow L^T \quad , \quad \mathcal{C} \leftrightarrow \mathcal{O}^T$$

The **observer design problem is a dual of the state feedback design problem**. **Observer design by eigenvalue assignment** The poles of the

estimation error equation can be arbitrarily re-located if the **system is observable** and then

$$L = \mathcal{O}^{-1} T_\ell^{-T} (\underline{\alpha} - \underline{a})$$

$$T_\ell = \begin{bmatrix} 1 & a_1 & a_2 & . & . & . & a_{n-1} \\ 0 & 1 & a_1 & . & . & . & a_{n-2} \\ 0 & 0 & 1 & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & 1 \end{bmatrix} \quad , \quad T_\ell^T = \mathcal{O}_o^{-1}$$

Control using estimated state

Problem statement *Given*:

- a SISO state-space model with parameters (A, B, C)
- an observer giving an estimate $\hat{x}$ using an observer gain L
- a **state feedback** $u = -k\hat{x} + v$

Controller: a **dynamic output feedback** controller

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - Bk - LC)\hat{x} + Ly$$
$$u = -k\hat{x} + v$$

consists of a controller-observer pair *Determine*: the gains k and L such that the closed-loop system fulsills the control goal **Separation principle**: the observer gain L and the feedback gain k can be designed separately

Pole-placement design of the controller-observer pair

Control aim:

- the observer is stable ($\alpha_o(s)$ is given)
- the poles of the closed-loop system are equal to the prescribed values ($\alpha_c(s)$ is given)

A necessary and sufficient condition: **joint controllability and observability** The observer gain L and the feedback gain k are designed

separately (see separation principle) using the pole-placement design principle