Számítógépvezérelt szabályozások elmélete

Stabilitás

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Overview

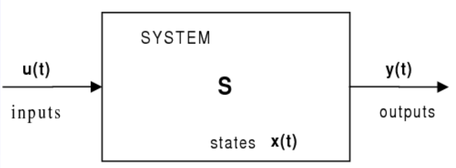
1. Basic notions
2. Bounded input-bounded output (BIBO) stability
3. Stability in the state space
4. Stability region of nonlinear systems

Systems

- System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

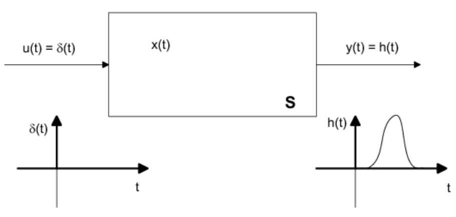
- inputs (u) and outputs (y)



CT-LTI I/O system models

- Time domain: Impulse response function
  is the response of a SISO LTI system to a Dirac-delta input function with zero initial condition.
- The output of **S** can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$



CT-LTI state-space models

- General form - revisited

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_0) = x(0)$$
$$y(t) = Cx(t)$$

with

  - signals: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^r$
  - system parameters: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{p \times n}$ $(D = 0)$
- Dynamic system properties:
  - observability ✓
  - controllability ✓
  - stability

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Signal spaces

- L_q spaces

$$L_q[0,\infty[= \left\{ f : \mathbb{R}_0^+ \mapsto \mathbb{R} | f \text{ is measurable and } \int_0^{\infty} |f(t)|^q < \infty \right\}$$

special case

$$L_\infty[0,\infty[= \left\{ f : \mathbb{R}_0^+ \mapsto \mathbb{R} | f \text{ is measurable and } \sup_{t\in\mathbb{R}_0^+} |f(t)| < \infty \right\}$$

- Remark: L_q spaces are Banach spaces with norms

$$||f||_q = \left(\int_0^{\infty} |f(t)|^q \right)^{1/q}$$

$$||f||_\infty = \sup_{t\in\mathbb{R}_0^+} |f(t)|$$

Vector valued signals

- $f(t) \in \nu$, $\forall t \geq 0$, where ν is a finite dimensional linear space (e.g. $\nu = \mathbb{R}^n$)
- $||\cdot||_\nu$ is a norm in ν (e.g. Euclidean)

$$L_q(\nu) = \left\{ f : \mathbb{R}_0^+ \mapsto \nu | f \text{ is measurable and } \int_0^\infty ||f(t)||_\nu^q < \infty \right\}$$

$$\text{norm: } ||f||_q = \left(\int_0^\infty ||f(t)||_\nu^q \right)^{\frac{1}{q}}$$

- Remark: The case L_2 is special, because the norm comes from an inner product (L_2 is a Hilbert-space)

BIBO stability – general

Definition

A system is externally or BIBO stable if for any bounded input it responds with a bounded output

$$||u|| \leq M_1 < \infty \Rightarrow ||y|| \leq M_2 < \infty$$

where $||.||$ is a signal norm.

- This applies to any type of systems.
- Stability is a system property, i.e. it is realization-independent.

BIBO stability – 1

- Bounded input-bounded output (BIBO) stability for SISO systems

$$|u(t)| \leq M_1 < \infty, \ \forall t \in [0, \infty[\ \Rightarrow \ |y(t)| \leq M_2 < \infty, \ \forall t \in [0, \infty[$$

Theorem (BIBO stability)

A SISO LTI system is BIBO stable if and only if

$$\int_0^\infty |h(t)| dt \leq M < \infty$$

where $M \in \mathbb{R}^+$ and $h(t)$ is the impulse response function.

- Question: How can we generalize for MIMO systems?

BIBO stability – 2

Proof.

- $\Leftarrow$ Assume $\int_0^\infty |h(t)|dt \leq M < \infty$ and u is bounded, i.e. $|u(t)| \leq M_1 < \infty$, $\forall t \in \mathbb{R}_0^+$. Then

$$|y(t)| \leq |\int_0^\infty h(\tau)u(t-\tau)d\tau| \leq M_1 \int_0^\infty |h(\tau)|d\tau \leq M_1 \cdot M = M_2$$

- $\Rightarrow$ (indirect) Assume $\int_0^\infty |h(\tau)|d\tau = \infty$, but the system is BIBO stable. Consider the **bounded** input:

$$u(t-\tau) = \text{sign } h(\tau) = \begin{cases} 1 & \text{if} \quad h(\tau) > 0 \\ 0 & \text{if} \quad h(\tau) = 0 \\ -1 & \text{if} \quad h(\tau) < 0 \end{cases}$$

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- Stability of nonlinear systems
- Asymptotic stability of CT-LTI systems
- The Lyapunov method

4 Stability region of nonlinear systems

Stability of nonlinear systems

- Consider the autonomous nonlinear system:

$$\dot{x} = f(x), \quad x \in \mathcal{X} = \mathbb{R}^n, \; f : \mathbb{R}^n \mapsto \mathbb{R}^n$$

with an equilibrium point: $f(x^*) = 0$

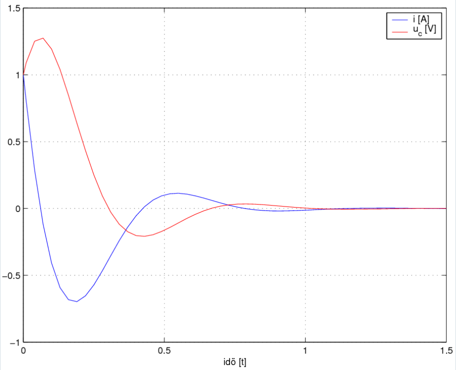
  - x^* stable equilibrium point: for any $\epsilon > 0$ there exists $\delta > 0$ $(\delta < \epsilon)$ such that for $\|x^* - x(0)\| < \delta$ $\|x^* - x(t)\| < \epsilon$ holds.
  - x^* asymptotically stable equilibrium point: x^* stable and $\lim_{t\to\infty} x(t) = x^*$.
  - x^* unstable equilibrium point: not stable
  - x^* locally (asymptotically) stable: there exists a neighborhood U of x^* within which the (asymptotic) stability conditions hold
  - x^* globally (asymptotically) stable: $U = \mathbb{R}^n$

Asymptotic stability

Example (Asymptotically stable system)

RLC circuit with parameters: $R = 1\ \Omega$, $L = 10^{-1} H$, $C = 10^{-1} F$.

$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V

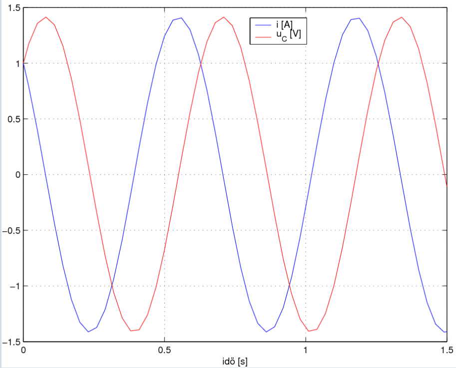


Non-asymptotic stability

Example (Non-asymptotically stable system)

RLC circuit with parameters: $R = 0\ \Omega,\ L = 10^{-1}H,\ C = 10^{-1}F.$

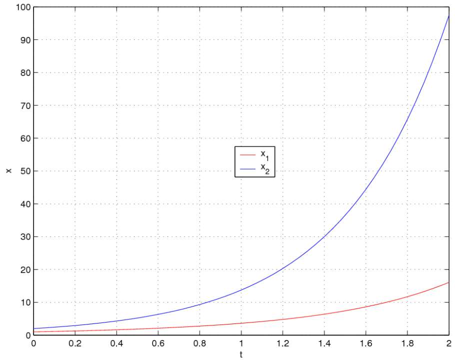
$u_C(0) = 1$ V, $i(0) = 1$ A, $u_{be}(t) = 0$ V



Instability

Example (Unstable system)

$$\begin{aligned}\dot{x}_1 &= x_1 + 0.1x_2 \\ \dot{x}_2 &= -0.2x_1 + 2x_2\end{aligned}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



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Stability of CT-LTI systems

- Truncated state equation with $u \equiv 0$:

$$\dot{x} = A \cdot x, \quad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n \times n}, \; x(0) = x_0$$

- Equilibrium point: $x^* = 0$
- Solution:

$$x(t) = e^{At} \cdot x_0$$

- Recall: A can be diagonalized (there exists invertible T, such that

$$T \cdot A \cdot T^{-1}$$

is diagonal) if and only if, A has n linearly independent eigenvectors.

Asymptotic stability of LTI systems – 1

Stability types:

- the real part of every eigenvalue of A is negative (A is a *stability matrix*): asymptotic stability
- A has eigenvalues with zero and negative real parts
  - the eigenvectors related to the zero real part eigenvalues are linearly independent: (non-asymptotic) stability
  - the eigenvectors related to the zero real part eigenvalues are not linearly independent: (polynomial) instability
- A has (at least) an eigenvalue with positive real part: (exponential) instability

Asymptotic stability of LTI systems – 2

Theorem

The eigenvalues of a square $A \in \mathbb{R}^{nxn}$ matrix remain unchanged after a similarity transformation on A by a transformation matrix T:

$$A' = TAT^{-1}$$

Proof.

Let us start with the eigenvalue equation for matrix A

$$A\xi = \lambda\xi \ , \ \xi \in \mathbb{R}^n \ , \ \lambda \in \mathbb{C}$$

If we transform it using $\xi' = T\xi$ then we obtain

$$TAT^{-1}T\xi = \lambda T\xi$$

$$A'\xi' = \lambda\xi'$$

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Asymptotic stability of LTI systems – 3

Theorem

A CT-LTI system is asymptotically stable iff A is a stability matrix.

Sketch of the Proof.

Assume, that A is diagonalizable

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

$$\bar{x}(t) = e^{\bar{A}t} \cdot \bar{x}_0 \quad , \quad e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ & & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}$$

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BIBO and asymptotic stability

Theorem

Asymptotic stability implies BIBO stability for LTI systems.

Proof.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad y(t) = Cx(t)$$

$$\begin{aligned}
||x(t)|| &\leq ||e^{At}x(t_0) + M\int_0^t e^{A(t-\tau)}Bd\tau|| \\
&= ||e^{At}(x(t_0) + M\int_0^t e^{-A\tau}Bd\tau)|| \\
&= ||e^{At}(x(t_0) + M[-A^{-1}e^{-A\tau}B]_0^t)|| \\
&= ||e^{At}(x(t_0) - MA^{-1}e^{-At}B + MA^{-1}B)]||
\end{aligned}$$

$$||x(t)|| \leq ||e^{At}(x(t_0) + MA^{-1}B) - MA^{-1}B||$$

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Lyapunov theorem of stability

- Lyapunov-function: $V : \mathcal{X} \mapsto \mathbb{R}$
  - $V > 0$, if $x \neq x^*$, $V(x^*) = 0$
  - V is continuously differentiable
  - V is non-increasing, i.e.

$$\frac{d}{dt}V(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) \leq 0$$

Theorem (Lyapunov stability theorem)

- *If there exists a Lyapunov function to the system $\dot{x} = f(x)$, $f(x^*) = 0$, then x^* is a stable equilibrium point.*
- *If $\frac{d}{dt}V \lneqq 0$ then x^* is an asymptotically stable equilibrium point.*
- *If the properties of a Lyapunov function hold only in a neighborhood U of x^*, then x^* is a locally (asymptotically) stable equilibrium point.*

Lyapunov theorem – example

Example

- System:

$$\dot{x} = -(x-1)^3$$

- Equilibrium point: $x^* = 1$
- Lyapunov function: $V(x) = (x-1)^2$

$$\frac{d}{dt}V = \frac{\partial V}{\partial x}\dot{x} = 2(x-1)\cdot(-(x-1)^3) =$$
$$= -2(x-1)^4 < 0$$

- The system is globally asymptotically stable

CT-LTI Lyapunov theorem – 1

Basic notions:

- $Q \in \mathbb{R}^{n \times n}$ symmetric matrix: $Q = Q^T$, i.e. $[Q]_{ij} = [Q]_{ji}$ (every eigenvalue of Q is real)
- symmetric matrix Q is positive definite ($Q > 0$): $x^T Q x > 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is positive)
- symmetric matrix Q is negative definite $Q < 0$: $x^T Q x < 0, \forall x \in \mathbb{R}^n$, $x \neq 0$ ($\Leftrightarrow$ every eigenvalue of Q is negative)

Theorem (Lyapunov criterion for LTI systems)

The state matrix (A) of an LTI system is a stability matrix if and only if there exists a positive definite symmetric matrix P for every given positive definite symmetric matrix Q such that

$$A^T P + PA = -Q$$

CT-LTI Lyapunov theorem – 2

Proof.

$\Leftarrow$ Assume $\forall\ Q > 0\ \exists\ P > 0$ such that $A^T P + PA = -Q$.
Let $V(x) = x^T Px$.

$$\frac{d}{dt}V = \dot{x}^T Px + x^T P\dot{x} = x^T(A^T P + PA)x < 0$$

$\Rightarrow$ Assume A is a stability matrix. Then

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

$$A^T P + PA = \int_0^\infty A^T e^{A^T t} Q e^{At} dt + \int_0^\infty e^{A^T t} Q e^{At} A dt = \left[e^{A^T t} Q e^{At} \right]_0^\infty = -Q$$

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Quadratic stability region

- Use quadratic Lyapunov function candidate with a given positive definite diagonal weighting matrix Q (tuning parameter!)

$$V\left[x(t)\right] = (x - x^*)^T \cdot Q \cdot (x - x^*)$$

- Dissipativity condition gives a conservative estimate of the stability region

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}\overline{f}(x)$$

  - $\overline{f}(x) = f(x)$ in the open loop case with $u = 0$
  - $\overline{f}(x) = f(x) + g(x) \cdot C(x)$ in the closed-loop case where $C(x)$ is the static state feedback

Quadratic stability region

Example

- Nonlinear system

$$\begin{aligned} \dot{x}_1 &= 0.4x_1x_2 - 1.5x_1 \\ \dot{x}_2 &= -0.8x_1x_2 - 1.5x_2 + 1.5u \\ y &= x_2 \end{aligned}$$

- Equilibrium point with $u^* = 7.75$

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 2 \\ 3.75 \end{bmatrix}$$

- Locally linearized system

$$\begin{aligned} \dot{\tilde{x}} &= \begin{bmatrix} 0 & 0.8 \\ -3 & -3.1 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \tilde{u} \\ \tilde{y} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{x} \end{aligned}$$

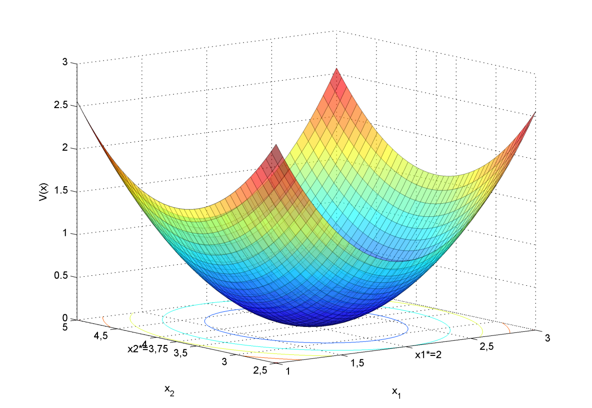
- Eigenvalues of the state matrix are $\lambda_1 = -1.5$ and $\lambda_2 = -1.6$ so equilibrium point x^* (not the system !) is locally asymptotically stable.

Quadratic stability region

Example (continued)

- Quadratic Lyapunov function

$$V(x) = (x - x^*)^T \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot (x - x^*)$$



Quadratic stability region

Example (continued)

- Time derivative of the quadratic Lyapunov function