Computer Controlled Systems

Sampling
Discrete time LTI models

Katalin Hangos

University of Pannonia
Faculty of Information Technology
Department of Electrical Engineering and Information Systems

hangos.katalin@virt.uni-pannon.hu

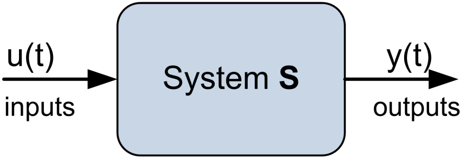
March 2017

Systems

System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

- inputs (u) and outputs (y)



CT-LTI system models

Input-output (I/O) models for SISO systems

- time domain
- operator domain

State-space models

CT-LTI I/O system models (SISO)

Transfer function – Linear diff. equation model

$$\mathcal{L}\{a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y\} =$$
$$= \mathcal{L}\{b_0 u + b_1 \frac{du}{dt} + \ldots + b_m \frac{d^m u}{dt^m}\}$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)}$$

Transfer function – Impulse response function

$$H(s) = \mathcal{L}\{h(t)\}$$

CT-LTI state-space models

General form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad & (\textit{state equation}) \\ y(t) &= Cx(t) + Du(t) \qquad & (\textit{output equation}) \end{aligned}$$

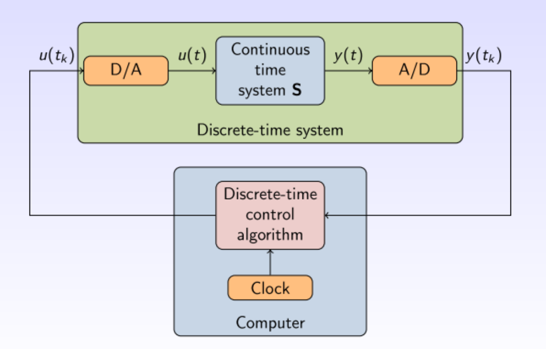
with

- given initial condition $x(t_0) = x(0)$ and $x(t) \in \mathcal{R}^n$,
- $y(t) \in \mathcal{R}^p$, $u(t) \in \mathcal{R}^r$
- system parameters

$$A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{p \times n} \ , \ D \in \mathbb{R}^{p \times r}$$

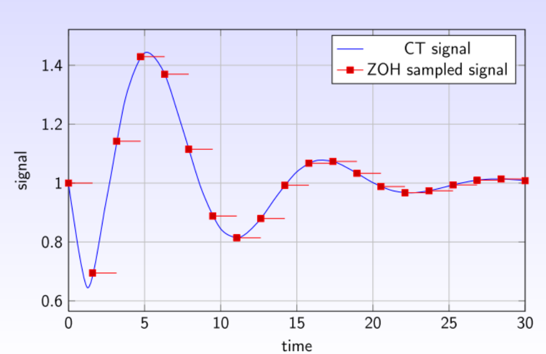
Sampling

System elements for sampling



Zero order hold sampling

Operation of the **D/A converter**



Sampling of CT-LTI systems

Given:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Zero order hold sampling of u

$$u(\tau) = u(t_k) = u(k) \quad , \quad t_k \leq \tau < t_{k+1}$$

Equidistant (periodic) **sampling**: $t_{k+1} - t_k = h = const$

Compute:
the state-space model of the sampled (discrete time) system

Sampled state equations - 1

Use the solution of the continuous time state equation

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Substitute $t = t_{k+1}$ and $t_0 = t_k$ with *periodic sampling* ($h = (t_{k+1} - t_k)$) and $\theta = \tau - t_k$.
Denote $x(k) = x(t_k)$ and $x(k+1) = x(t_{k+1})$

$$x(k+1) = e^{Ah}x(k) + e^{Ah}\int_{0}^{h} e^{-A\theta}d\theta Bu(k)$$

Discrete time state equation

$$x(k+1) = e^{Ah}x(k) + A^{-1}(e^{Ah} - I)Bu(k)$$

Sampled state equations - 2

DT-LTI state equation for **sampled** systems

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$\Phi = e^{Ah} = I + Ah + ... \quad , \quad \Gamma = A^{-1}(e^{Ah} - I)B = (Ih + \frac{Ah^2}{2!} + ...)B$$

Matrix exponential function

Given $A \in \mathbb{R}^{n \times n}$ and the real-valued exponential function $e \ : \ \mathbb{R} \mapsto \mathbb{R}$

Take the Taylor-series expansion of e around $t = 0$

$$e^t = 1 + t + \frac{1}{2}t^2 + ... + \frac{1}{j!}t^j + ...$$

Substitute $t = A$ and $1 = I$

$$e^A = I + A + \frac{1}{2}A^2 + ... + \frac{1}{j!}A^j + ... \quad \in \mathbb{R}^{n \times n}$$

DT-LTI state-space models

$$x(k+1) = \Phi x(k) + \Gamma u(k) \qquad (state\ equation)$$
$$y(k) = Cx(k) + Du(k) \qquad (output\ equation)$$

with given initial condition $x(0)$ and

$$x(k) \in \mathbb{R}^n \ , \ y(k) \in \mathbb{R}^p \ , \ u(k) \in \mathbb{R}^r$$

being vectors of finite dimensional spaces and

$$\Phi \in \mathbb{R}^{n \times n} \ , \ \Gamma \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{p \times n} \ , \ D \in \mathbb{R}^{p \times r}$$

being matrices

Transformation of states

$$x(k+1) = \Phi x(k) + \Gamma u(k) \quad , \quad \overline{x}(k+1) = \overline{\Phi}\overline{x}(k) + \overline{\Gamma} u(k)$$
$$y(k) = Cx(k) + Du(k) \quad , \quad y(k) = \overline{C}\overline{x}(k) + \overline{D}u(k)$$

which are related by the transformation

$$T \in \mathbb{R}^{n \times n} \quad , \quad \det\ T \neq 0 \quad , \quad \overline{x} = Tx \quad \Rightarrow \quad x = T^{-1}\overline{x}$$

$$dim\ \mathcal{X} = dim\ \overline{\mathcal{X}} = n$$

$$T^{-1}\overline{x}(k+1) = \Phi T^{-1}\overline{x}(k) + \Gamma u(k)$$

$$\overline{x}(k+1) = T\Phi T^{-1}\overline{x}(k) + T\Gamma u(k) \quad , \quad y(k) = CT^{-1}\overline{x}(k) + Du(k)$$

$$\overline{\Phi} = T\Phi T^{-1} \quad , \quad \overline{\Gamma} = T\Gamma \quad , \quad \overline{C} = CT^{-1} \quad , \quad \overline{D} = D$$

Solution of the DT-LTI state equation

$$
\begin{aligned}
&x(1) = \Phi x(0) + \Gamma u(0) \\
&x(2) = \Phi x(1) + \Gamma u(1) = \Phi^2 x(0) + \Phi\Gamma u(0) + \Gamma u(1) \\
&x(3) = \Phi x(2) + \Gamma u(2) = \Phi^3 x(0) + \Phi^2\Gamma u(0) + \Phi\Gamma u(1) + \Gamma u(2) \\
&.. \\
&.. \\
&x(k) = \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1}\Gamma u(j)
\end{aligned}
$$

DT-LTI SISO I/O system models – Pulse response function

From the solution of the state equation with $D = 0$

$$x(k) = \Phi x(k-1) + \Gamma u(k-1) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1} \Gamma u(j)$$
$$y(k) = Cx(k) = C\Phi^k x(0) + \sum_{j=0}^{k-1} C\Phi^{k-j-1} \Gamma u(j)$$

Pulse response function

$$h(k) = \begin{cases} 0 & k < 1 \\ C\Phi^{k-1}\Gamma & k \geq 1 \end{cases}$$

The *discrete time analogue of the impulse response function*.

Discrete time Markov parameters: $C\Phi^{k-1}\Gamma$

- they are invariant for the state LTI transformations ($\overline{x} = Tx$)

Discrete time signals

$$f = \{f(k), k = 0, 1, ...\}$$

Signal norms for *scalar valued discrete time signals*

- the *infinity norm*

$$||f||_\infty = \sup_k |f(k)|$$

- the *2-norm*

$$||f||_2^2 = \sum_{k=-\infty}^{\infty} f^2(k)$$

Shift operators

Definition (forward shift operator q)

which acts on a discrete time signal as follows

$$qf(k) = f(k+1) \tag{1}$$

Definition (backward shift operator (delay) q^{-1})

which acts on a discrete time signal as follows

$$q^{-1}f(k) = f(k-1) \tag{2}$$

- **The induced norm of an operator** q on the vector space X induced by a norm $||.||$ on the same space is defined as

$$||q|| = \sup_{||x||=1} \frac{||q(x)||}{||x||}$$

DT-LTI SISO I/O system models – Discrete difference equation models

- **Forward difference form** with $n_a \geq n_b$ (proper)

  $$y(k+n_a)+a_1 y(k+n_a-1)+...+a_{n_a} y(k) = b_0 u(k+n_b)+...+b_{n_b} u(k)$$

  Compact form

  $$A(q)y(k) = B(q)u(k)$$

  $$A(q) = q^{n_a} + a_1 q^{n_a-1} + ... + a_{n_a} \ , \ B(q) = b_0 q^{n_b} + b_1 q^{n_b-1} + ... + b_{n_b}$$

- **Backward difference form** where $d = n_a - n_b > 0$ is the *pole excess (time delay)*

  $$y(k)+a_1 y(k-1)+...+a_{n_a} y(k-n_a) = b_0 u(k-d)+...+b_{n_b} u(k-d-n_b)$$

  $$A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d) \ , \ A^*(q^{-1}) = q^{n_a} A(q^{-1}), \ B^*(q^{-1}) = q$$

DT-LTI SISO I/O system models – Pulse transfer operator

- Computed from the DT-LTI state-space model

$$x(k+1) = \Phi x(k) + \Gamma u(k) \quad , \quad y(k) = Cx(k) + Du(k)$$

$$\begin{aligned} x(k+1) &= qx(k) = \Phi x(k) + \Gamma u(k) \\ x(k) &= (qI - \Phi)^{-1}\Gamma u(k) \\ y(k) &= Cx(k) + Du(k) = [C(qI - \Phi)^{-1}\Gamma + D]u(k) \end{aligned}$$

Pulse-transfer operator $H(q)$ of the SSR (Φ, Γ, C, D):

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D$$

The *discrete time analogue of the transfer function.*

DT-LTI SISO I/O system models – Pulse transfer operator

- For SISO LTI systems $H(q)$ is a rational function

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D = \frac{B(q)}{A(q)} \quad , \quad deg \; B(q) < deg \; A(q) = n$$

  where $A(q)$ is the characteristic polynomial of the state matrix Φ.
- Relation with the **discrete difference equation form**

$$\begin{aligned} & y(k+n_a) + a_1 y(k+n_a-1) + \ldots + a_{n_a} y(k) = \\ & \quad = b_0 u(k+n_b) + \ldots + b_{n_b} u(k) \end{aligned}$$

$$A(q)y(k) = B(q)u(k)$$

Poles of DT-LTI systems – 1

- Comparison

	continuous time system	discrete time system
state eq.	$\dot{x}(t) = Ax(t) + Bu(t)$	$x(k+1) = \Phi x(k) + \Gamma u(k)$ $\Phi = e^{Ah}$
output eq.	$y(t) = Cx(t)$	$y(k) = Cx(k)$
poles	$\lambda_i(A)$	$\lambda_i(\Phi)$ $\lambda_i(\Phi) = e^{\lambda_i(A)h}$

Poles of DT-LTI systems – 2