Computer Controlled Systems

Signals and systems
Construction of state space models

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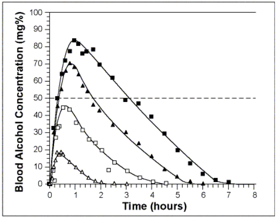
Overview

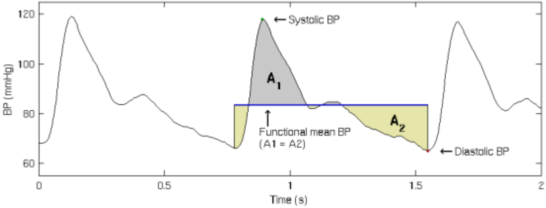
Signals – 1

Signal:
time-varying (and/or spatial varying) quantity

Examples

- $x : \mathbb{R}_0^+ \mapsto \mathbb{R}, \quad x(t) = e^{-t}$
- $y : \mathbb{N}_0^+ \mapsto \mathbb{R}, \quad y[n] = e^{-n}$
- $X : \mathbb{C} \mapsto \mathbb{C}, \quad X(s) = \frac{1}{s+1}$

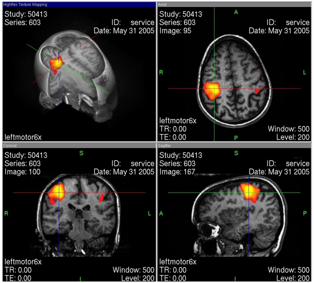


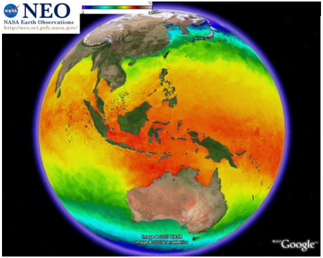


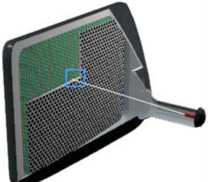
Signals – 2

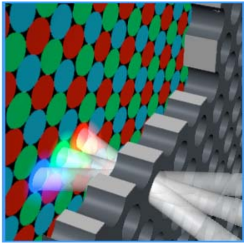
- surface temperature $T(r, \theta, \phi, t)$ on Earth: $T : \mathbb{R}^+ \times [0, \pi] \times [0, 2\pi] \mapsto \mathbb{R}$ (r, θ, ϕ: spherical coordinates, t: time)
- colored TV screen: $I : \mathbb{N}^3 \mapsto \mathbb{N}^3$

$$I(x, y, t) = \begin{bmatrix} I_R(x, y, t) \\ I_G(x, y, t) \\ I_B(x, y, t), \end{bmatrix}$$









Classification of signals

- dimension of the independent variable - **only time-dependent** vs. other dependencies
- dimension of the signal - **scalar vs. vector-valued**
- **real-valued** vs. complex-valued
- **continuous time vs. discrete time**
- **continuous valued** vs. discrete valued
- bounded vs. unbounded
- periodic vs. aperiodic
- even vs. odd

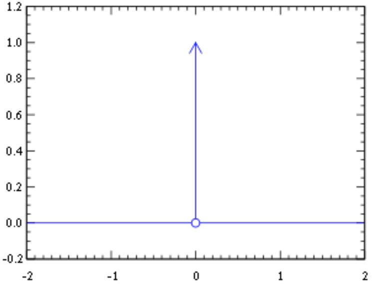
Special signals – 1

Dirac-δ or unit impulse function

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

where $f : \mathbb{R}_0^+ \mapsto \mathbb{R}$ arbitrary smooth (many times continuously differentiable) function.
Consequence:

$$\int_{-\infty}^{\infty} 1 \cdot \delta(t)dt = 1$$



Physical meaning of the unit impulse:

- force impulse $\Rightarrow$ momentum
- density impulse $\Rightarrow$ mass point

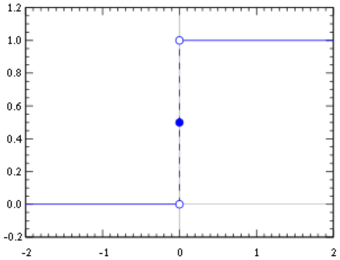
Special signals – 2

Unit step function

$$\eta(t) = \int_{-\infty}^{t} \delta(\tau) d\tau,$$

i.e.

$$\eta(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$$

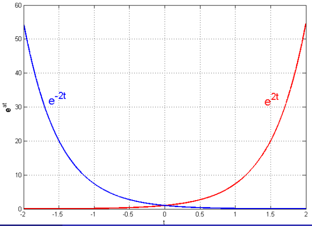


Exponential function

$$e^{at}, \quad a \in \mathbb{R}$$

Complex exponential: $a \in \mathbb{C}, \; a = \alpha + \mathrm{j}\Omega$

$$e^{at} = e^{\alpha t} \cdot e^{\mathrm{j}\Omega t} = e^{\alpha t}\cos(\Omega t) + \mathrm{j}e^{\alpha t}\sin(\Omega t)$$



Basic operations on signals – 1

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

- addition:
  $(x + y)(t) = x(t) + y(t), \quad \forall t \in \mathbb{R}_0^+$
- multiplication by scalar:
  $(\alpha x)(t) = \alpha x(t) \quad \forall t \in \mathbb{R}_0^+, \ \alpha \in \mathbb{R}$
- scalar product:
  $\langle x, y\rangle(t) = \langle x(t), y(t)\rangle \quad \forall t \in \mathbb{R}_0^+$

Basic operations on signals – 2

- time shift:
  $\mathbf{T}_a x(t) = x(t-a) \quad \forall t \in \mathbb{R}_0^+, a \in \mathbb{R}$
- convolution: $x, y : \mathbb{R}_0^+ \mapsto \mathbb{R}$

$$(x * y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau, \quad \forall t \geq 0$$

Laplace-transformation

Domain:

$$\Lambda = \{\ f \mid f : \mathbb{R}_0^+ \mapsto \mathbb{C}, f \text{ integrable on } [0, a],\ \forall a > 0 \text{ and}$$
$$\exists A_f \geq 0, a_f \in \mathbb{R}, \text{ such that } |f(x)| \leq A_f e^{a_f x}\ \forall x \geq 0\ \}$$

Laplace-transform (connection with Fourier transform: $s = \mathrm{j}\Omega$)

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \quad f \in \Lambda,\ s \in \mathbb{C}, \quad s = \sigma + \mathrm{j}\Omega$$

Properties

- Linear: $\mathcal{L}\{c_1 y_1 + c_2 y_2\} = c_1 \mathcal{L}\{y_1\} + c_2 \mathcal{L}\{y_2\}$
- $\mathcal{L}\{\frac{dy}{dt}\} = sY(s) - y(0)$
- $\mathcal{L}\{\int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau\} = H(s)U(s)$

Inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi \mathrm{j}} \int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} F(s)e^{st}ds, \quad t \in \mathbb{R}_0^+$$

Overview

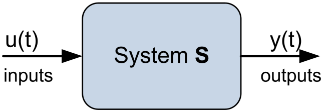
1. Signals
2. Systems
   - System properties
   - System model types
3. Construction of state-space models

Systems

System ($\mathbf{S}$): acts on signals

$$y = \mathbf{S}[u]$$

- inputs ($u \in \mathcal{U}$) and outputs ($y \in \mathcal{Y}$)
- abstract operator ($\mathbf{S} : \mathcal{U} \to \mathcal{Y}$)



Basic system properties – 1

- Linearity

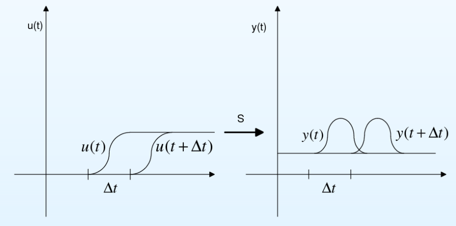
$$\mathbf{S}[c_1 u_1 + c_2 u_2] = c_1 y_1 + c_2 y_2$$

with $c_1, c_2 \in \mathbb{R}, \quad u_1, u_2 \in \mathcal{U}, \quad y_1, y_2 \in \mathcal{Y}$ and $\mathbf{S}[u_1] = y_1, \quad \mathbf{S}[u_2] = y_2$
Linearity check: use the definition

- Time-invariance

$$\mathbf{T}_\tau \circ \mathbf{S} = \mathbf{S} \circ \mathbf{T}_\tau$$

where $\mathbf{T}_\tau$ is the time-shift operator: $\mathbf{T}_\tau(u(t)) = u(t+\tau), \quad \forall t$
Time invariance check: **constant parameters**



Basic system properties – 2

- SISO/MIMO
  Single Input-Single Output, or Multiple Input-Multiple Output sytems
- Continuous-time (CT) and Discrete-time (DT) systems
  Continuous-time system: the time set $\mathcal{T} \subseteq \mathbb{R}$
  Discrete-time system: the time set $\mathcal{T} = \{\ldots, t_{-1}, t_0, t_1, t_2, \ldots\}$
- Causality
  The present does not depend on the future, only on the past.

System model types

- Input-output (I/O) models (*for SISO systems in this course*)
  - time domain
  - frequency domain
  - operator domain
- State-space models

State-space models

General form

$$\begin{array}{ll} \dot{x}(t) = \mathcal{F}(x(t), u(t)) & \text{(state equation)} \\ y(t) = \mathcal{H}(x(t), u(t)) & \text{(output equation)} \end{array} , \quad x(t_0) = x_0$$

with

- given initial condition $x(t_0) = x_0$,
- $x(t) \in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^p, \quad u(t) \in \mathbb{R}^r$ – signals, time-dependent quantities
- state equation is a set of differential equations
- output equation is a set of algebraic equations in the MIMO case
- system parameters – constants, do not depend on time

Overview

Conservation balances

Balance volumes: for constructing conservation balances

- most often with *constant volume*
- *perfectly stirred* (concentrated parameter, the balance is in the form of ordinary differential equations)

Conserved (extensive) **quantities**:

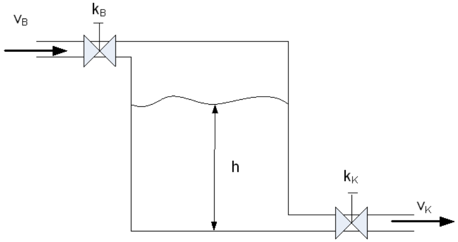
- ovarall mass
- energy (entalpy, internal energy)
- component mass, (momentum)

Dynamic conservation balance in general form: for a conserved quantity

$$\left\{ \begin{array}{c} \textit{rate of} \\ \textit{change} \end{array} \right\} = \left\{ \begin{array}{c} \textit{in-} \\ \textit{flow} \end{array} \right\} - \left\{ \begin{array}{c} \textit{out-} \\ \textit{flow} \end{array} \right\} + \left\{ \begin{array}{c} \textit{source} \\ \textit{sink} \end{array} \right\}$$

Example: tank with gravitational outflow - 1

Problem description
Given a tank with constant cross section that is used for storing water. The water flows into the tank through a binary input valve, the outflow rate is driven by gravitation, i.e. depends on the water level in the tank, but it is controlled by a binary output valve.



Construct the model of the tank for diagnostic purposes if we can measure the water level and the status of the valves.

Example: tank with gravitational outflow - 2

Conservation balance equation: for overall mass

$$\frac{dm}{dt} = v_b - v_k \tag{1}$$

Constitutive equations

- $m = A \cdot h \cdot \rho$ (water level h is measurable)
- $v_B = v_B^* k_B$ (valve status k_B is measurable)
- $v_K = K \cdot h \cdot k_K$ (gravitational outflow, valve status k_K is measurable)

Example: tank with gravitational outflow - 3

Model equation with measurable variables:

$$\frac{dh}{dt} = \frac{v_b^*}{A\rho} k_b - \frac{K}{A\rho} h \cdot k_K \qquad (2)$$

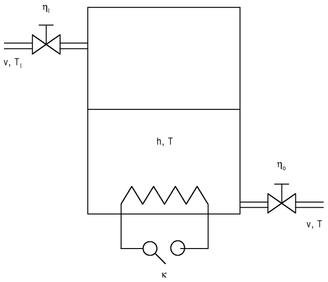
State-space model form

- state variable: water level h
- input variables: status of the valves k_B and k_K
- output variable: water level h

Example: Coffee machine - 1

Problem description

Given a tank with constant cross section equipped with an electric heater that is used for boiling water water. The water flows into the tank through a binary input valve, and the outflow is also controlled by a binary output valve. The heater is controlled by a binary switch.



Construct the model of the coffee machine if we can measure the water level, the water temperature and the status of the valves and the switch.

Example: Coffee machine - 2

Conservation balance equation: for overall mass

$$\frac{dM}{dt} = \rho v_I - \rho v_O \tag{3}$$

Conservation balance equation: for internal energy

$$\frac{dE}{dt} = c_P \rho T_I v_I - c_P \rho T v_O + \kappa H \tag{4}$$

Constitutive equations

$$M = \rho A h \tag{5}$$

$$E = c_P \rho A h T \tag{6}$$

$$v_I = \eta_I v \quad , \quad v_O = \eta_O v \tag{7}$$

Example: Coffee machine - 3

Model equation with measurable variables:

$$\frac{dh}{dt} = \frac{1}{A}\eta_I v - \frac{1}{A}\eta_O v \tag{8}$$

$$\frac{dT}{dt} = \frac{1}{A}\eta_I v T_I \frac{1}{h} - \frac{1}{A}\eta_O v T \frac{1}{h} + \frac{H}{c_P \rho A}\kappa \frac{1}{h} \tag{9}$$

State-space model form

- state variables: water level h, temperature T
- input variables: status of the valves η_I and η_O, switch κ, inlet temperature T_I
- output variable: water level h, temperature T

Parameters: A, H, c_P, ρ, v