# Discrete and Continuous Dynamical Systems

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# Overview

# Transformation of states

Two different state space models with the same input-output behavior

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \quad , \quad \dot{\overline{\boldsymbol{x}}}(t) = \overline{\boldsymbol{A}}\overline{\boldsymbol{x}}(t) + \overline{\boldsymbol{B}}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \quad , \quad \boldsymbol{y}(t) = \overline{\boldsymbol{C}}\overline{\boldsymbol{x}}(t) + \overline{\boldsymbol{D}}\boldsymbol{u}(t) \end{aligned}$$

which are related by the transformation

$$\boldsymbol{T} \in \mathbb{R}^{n\times n} \quad , \quad det\ \boldsymbol{T} \neq 0 \quad , \quad \overline{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x} \quad \Rightarrow \quad \boldsymbol{x} = \boldsymbol{T}^{-1}\overline{\boldsymbol{x}}$$

$$dim\ \mathcal{X} = dim\ \overline{\mathcal{X}} = n$$

$$\boldsymbol{T}^{-1}\dot{\overline{\boldsymbol{x}}} = \boldsymbol{A}\boldsymbol{T}^{-1}\overline{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u}$$

$$\dot{\overline{\boldsymbol{x}}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\overline{\boldsymbol{x}} + \boldsymbol{T}\boldsymbol{B}\boldsymbol{u} \quad , \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{T}^{-1}\overline{\boldsymbol{x}} + \boldsymbol{D}\boldsymbol{u}$$

$$\overline{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} \quad , \quad \overline{\boldsymbol{B}} = \boldsymbol{T}\boldsymbol{B} \quad , \quad \overline{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{T}^{-1} \quad , \quad \overline{\boldsymbol{D}} = \boldsymbol{D}$$

# Overview

# Controllable canonical form (controller form)

- $H(s) = \frac{b(s)}{a(s)}$
- Controllability canonical form of the state space model

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -a_1 & \dots & -a_{n-1} & -a_n \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} \boldsymbol{x}(t)$$

- The change of the $i$-th state variable depends on the $i-1$-th one, $i > 1$
- The change of $x_1$ depends on all states and the input
- Always controllable

# Observable canonical form

- $H(s) = \frac{b(s)}{a(s)}$
- Observability canonical form of the state space model

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \boldsymbol{x}(t)$$

- Each state variable is fed back to the previous one and the output of the system is $x_1$.
- Always observable

# Diagonal form (or modal form) realization

- State space model in diagonal form

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_D\boldsymbol{x}(t) + \boldsymbol{B}_D u(t)\\ y(t) &= \boldsymbol{C}_D\boldsymbol{x}(t)\end{aligned}$$

with

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \lambda_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & \lambda_n \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & . & . & . & c_n \end{bmatrix}\boldsymbol{x}$$

# Controllability in diagonal form realization

- Controllability matrix

$$\mathcal{C}_n = \left[\begin{array}{ccccc} \boldsymbol{B} & \boldsymbol{AB} & . & . & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{array}\right] = \left[\begin{array}{ccccc} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & . & . \end{array}\right] =$$

$$= \left[\begin{array}{ccccc} b_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & b_n \end{array}\right] \left[\begin{array}{ccccc} 1 & \lambda_1 & . & . & \lambda^{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \lambda_n & . & . & \lambda_n^{n-1} \end{array}\right]$$

- The last matrix is a Vandermonde matrix $V$ with determinant

$$det\ \boldsymbol{V} = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$$

- Full rank of the controllability matrix

$$rank\ \mathcal{C}_n = n \quad \Leftrightarrow \quad det\ \mathcal{C}_n = \prod_i b_i \prod_{j<i} (\lambda_i - \lambda_j) \neq 0$$

# Controllability and observability in diagonal form realization

**Theorem (Controllability)**

*DSSR is controllable* ***iff*** $\lambda_i \neq \lambda_j, (i \neq j)$ ***and*** $b_i \neq 0, \ \forall i$

**Theorem (Observability)**

*DSSR is observable* ***iff*** $\lambda_i \neq \lambda_j, (i \neq j)$ ***and*** $c_i \neq 0, \ \forall i$

## The transfer function of diagonal form realization

- Transfer function

$$H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)}$$

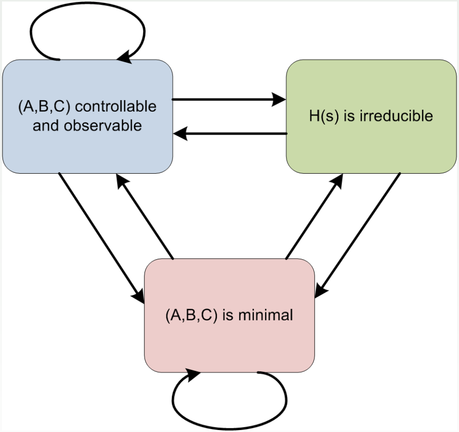
where $\boldsymbol{I}$ is a unit matrix.

- If either $c_j = 0$ or $b_k = 0$ then the transfer function can be described by smaller number of partial fractions than the original:

$$H(s) = \sum_{i=1}^{\overline{n}} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)} \quad , \quad \overline{n} < n$$

# Overview

# Equivalent SSR properties



# Overview

Consider **SISO CT-LTI systems** with realization $(A, B, C)$

- Joint controllability and observability is a **system property**
- Equivalent necessary and sufficient conditions
- Minimality of SSRs
- Irreducibility of the transfer function

# Hankel matrices

- Definition A Hankel matrix is a block matrix of the following form

$$\boldsymbol{H}[1, n-1] = \begin{bmatrix} \boldsymbol{CB} & \boldsymbol{CAB} & . \quad . \quad . & \boldsymbol{CA}^{n-1}\boldsymbol{B} \\ \boldsymbol{CAB} & \boldsymbol{CA}^2\boldsymbol{B} & . \quad . \quad . & \boldsymbol{CA}^n\boldsymbol{B} \\ . & . & . \quad . \quad . & . \\ . & . & . \quad . \quad . & . \\ . & . & . \quad . \quad . & . \\ \boldsymbol{CA}^{n-1}\boldsymbol{B} & \boldsymbol{CA}^n\boldsymbol{B} & . \quad . \quad . & \boldsymbol{CA}^{2n-2}\boldsymbol{B} \end{bmatrix}$$

- It contains *Markov parameters* $\boldsymbol{CA}^i\boldsymbol{B}$ that are invariant under state transformations.

# Lemma 1

## Lemma (1)

*If we have a system with transfer function* $H(s) = \frac{b(s)}{a(s)}$ *and there is an* $n$*-th order realization* $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$*, which is controllable and observable then all other* $n$*-th order realizations are controllable and observable.*

*Proof*

$$\mathcal{O}(\boldsymbol{C}, \boldsymbol{A}) = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}, \quad \mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} & . & . & . & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix}$$

$$\boldsymbol{H}[1, n-1] = \mathcal{O}(\boldsymbol{C}, \boldsymbol{A})\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$$

# Definitions

## Definition (Relative prime polynomials)

Two polynomials $a(s)$ and $b(s)$ are *coprime* (or relative primes) iff $a(s) = \prod(s - \alpha_i)$; $b(s) = \prod(s - \beta_j)$ and $\alpha_i \neq \beta_j$ for all $i, j$. In other words: the polynomials have no common factors.

## Definition (Irreducible transfer function)

A transfer function $H(s) = \frac{b(s)}{a(s)}$ is called to be irreducible if the polynomials $a(s)$ and $b(s)$ are relative primes.

# Lemma 2

## Lemma (2)

*If we have a controller form realization which is jointly controllable and observable then $a(s)$ and $b(s)$ are relative primes ($H(s)$ is irreducible).*

*Proof*

- A controller form realization is controllable and

$$\mathcal{O}_c = \tilde{\boldsymbol{I}}_n b(\boldsymbol{A_c})$$

$$\tilde{\boldsymbol{I}}_n = \begin{bmatrix} 0 & . & . & 1 \\ 0 & . & 1 & 0 \\ . & . & . & . \\ 1 & 0 & . & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

- Non-singularity of $b(\boldsymbol{A}_c)$

# Proof of Lemma 2

$$\tilde{\boldsymbol{I}}_n = \begin{bmatrix} \boldsymbol{e}_n & \boldsymbol{e}_{n-1} & . & . & \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_n^T \\ \boldsymbol{e}_{n-1}^T \\ . \\ . \\ . \\ \boldsymbol{e}_1^T \end{bmatrix} \quad , \quad \boldsymbol{e}_i = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \end{bmatrix} \leftarrow i.$$

$$\boldsymbol{A}_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix} \quad , \quad \boldsymbol{e}_i^T \boldsymbol{A}_c = \begin{cases} [-a_1 \quad -a_2 \quad \ldots \quad -a_n] \\ \boldsymbol{e}_{i-1}^T \end{cases}$$

# Proof of Lemma 2

- Computation of the observability matrix $\mathcal{O}_c = \tilde{I}_n b(\boldsymbol{A}_c) \in \mathbb{R}^{n \times n}$
- 1st row:

$$\boldsymbol{e}_n^T b(\boldsymbol{A}_c) = \boldsymbol{e}_n^T b_1 \boldsymbol{A}_c^{n-1} + ... + \boldsymbol{e}_n^T b_{n-1} \boldsymbol{A}_c + \boldsymbol{e}_n^T b_n \boldsymbol{I}_n$$

$n$-th term: $[0 \quad ... \quad 0 \quad b_n]$
$(n-1)$-th term: $b_{n-1} \boldsymbol{e}_n^T \boldsymbol{A}_c = b_{n-1} \boldsymbol{e}_{n-1}^T = [0 \quad ... \quad b_{n-1} \quad 0]$
...

$$\boldsymbol{e}_n^T b(\boldsymbol{A}_c) = [b_1 \quad ... \quad b_{n-1} \quad b_n] = C_c$$

- 2nd row:

$$\boldsymbol{e}_{n-1}^T b(\boldsymbol{A}_c) = \boldsymbol{e}_n^T \boldsymbol{A}_c b(\boldsymbol{A}_c) = \boldsymbol{e}_n^T b(\boldsymbol{A}_c) \boldsymbol{A}_c \quad \Rightarrow \quad \boldsymbol{e}_{n-1}^T b(\boldsymbol{A}_c) = \boldsymbol{C}_c \boldsymbol{A}_c$$

- and so on ...

# Proof of Lemma 2

$\mathcal{O}_c$ is nonsingular

- iff $b(\boldsymbol{A}_c)$ is nonsingular because matrix $\tilde{\boldsymbol{I}}_n$ is always nonsingular
- $b(\boldsymbol{A}_c)$ is nonsingular iff $det(b(\boldsymbol{A}_c)) \neq 0$
  which depends on the eigenvalues of $b(\boldsymbol{A}_c)$ matrix
- the eigenvalues of the matrix $b(\boldsymbol{A}_c)$ are $b(\lambda_i), \quad i = 1, 2, ..., n$
  $\lambda_i$ is an eigenvalue of $\boldsymbol{A}_c$, i.e a root of $a(s) = det(s\boldsymbol{I} - \boldsymbol{A})$

$$det(b(\boldsymbol{A}_c)) = \prod_{i=1}^{n} b(\lambda_i) \neq 0$$

$\Updownarrow$ $a(s)$ and $b(s)$ have no common roots, i.e. they are relative primes

# Minimal realization conditions

**Theorem (1)**

*$H(s) = \frac{b(s)}{a(s)}$ is irreducible iff all $n$-th order realizations are jointly controllable and observable.*

*Proof:* combine Lemma 1. and 2.

**Definition (Minimal realization)**

A realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ of dimension $n$ is minimal if one cannot find another realization of dimension less than $n$.

**Theorem (2)**

*$H(s) = \frac{b(s)}{a(s)}$ is irreducible iff any of its realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is minimal where $H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$*

*Proof:* by contradiction

# Minimal realization conditions

**Theorem (3)**

*A realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is minimal iff the system is jointly controllable and observable.*

*Proof:* Combine Theorem 1 and Theorem 2 .

**Lemma (3)**

*Any two minimal realizations can be connected by a unique similarity transformation (which is invertible).*

*Proof:* (Just the idea of it)

$$\boldsymbol{T} = \mathcal{O}^{-1}(\boldsymbol{C}_1, \boldsymbol{A}_1)\mathcal{O}(\boldsymbol{C}_2, \boldsymbol{A}_2) = \mathcal{C}(\boldsymbol{A}_1, \boldsymbol{B}_1)\mathcal{C}^{-1}(\boldsymbol{A}_2, \boldsymbol{B}_2)$$

exists and it is invertible: this is used as a transformation matrix.

# Overview

# General decomposition theorem

Given an $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ SSR, it is always possible to transform it to another realization $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{C}})$ with partitioned state vector and matrices

$$\overline{\boldsymbol{x}} = \begin{bmatrix} \overline{\boldsymbol{x}}_{co} & \overline{\boldsymbol{x}}_{c\bar{o}} & \overline{\boldsymbol{x}}_{\bar{c}o} & \overline{\boldsymbol{x}}_{\overline{co}} \end{bmatrix}^T$$

$$\overline{\boldsymbol{A}} = \begin{bmatrix} \overline{\boldsymbol{A}}_{co} & \mathbf{0} & \overline{\boldsymbol{A}}_{13} & \mathbf{0} \\ \overline{\boldsymbol{A}}_{21} & \overline{\boldsymbol{A}}_{c\bar{o}} & \overline{\boldsymbol{A}}_{23} & \overline{\boldsymbol{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & 0 & \overline{\boldsymbol{A}}_{43} & \overline{\boldsymbol{A}}_{\overline{co}} \end{bmatrix} \quad \overline{\boldsymbol{B}} = \begin{bmatrix} \overline{\boldsymbol{B}}_{co} \\ \overline{\boldsymbol{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\overline{\boldsymbol{C}} = \begin{bmatrix} \overline{\boldsymbol{C}}_{co} & \mathbf{0} & \overline{\boldsymbol{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix}$$

# General decomposition theorem

The partitioning defines subsystems

- *Controllable and observable subsystem*: $(\overline{\boldsymbol{A}}_{co}, \overline{\boldsymbol{B}}_{co}, \overline{\boldsymbol{C}}_{co})$ is minimal, i.e. $\overline{n} \leq n$ and

$$H(s) = \overline{\boldsymbol{C}}_{co}(s\overline{\boldsymbol{I}} - \overline{\boldsymbol{A}}_{co})^{-1}\overline{\boldsymbol{B}}_{co} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

- *Controllable subsystem*

$$\left( \begin{bmatrix} \overline{\boldsymbol{A}}_{co} & \mathbf{0} \\ \overline{\boldsymbol{A}}_{21} & \overline{\boldsymbol{A}}_{c\overline{o}} \end{bmatrix} , \begin{bmatrix} \overline{\boldsymbol{B}}_{co} \\ \overline{\boldsymbol{B}}_{c\overline{o}} \end{bmatrix} , \begin{bmatrix} \overline{\boldsymbol{C}}_{co} & \mathbf{0} \end{bmatrix} \right)$$

- *Observable subsystem*

$$\left( \begin{bmatrix} \overline{\boldsymbol{A}}_{co} & \overline{\boldsymbol{A}}_{13} \\ \mathbf{0} & \overline{\boldsymbol{A}}_{\overline{c}o} \end{bmatrix} , \begin{bmatrix} \overline{\boldsymbol{B}}_{co} \\ \mathbf{0} \end{bmatrix} , \begin{bmatrix} \overline{\boldsymbol{C}}_{co} & \overline{\boldsymbol{C}}_{\overline{c}o} \end{bmatrix} \right)$$

- *Uncontrollable and unobservable subsystem*

$$([\overline{\boldsymbol{A}}_{\overline{co}}] \ , \ \mathbf{0} \ , \ \mathbf{0})$$