Advanced parameter estimation

Design of experiments for parameter estimation

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Lecture overview

1. The parameter estimation problem - a repetition
   - Model types for parameter estimation
2. Advanced experiment design by using sensitivities
   - Parametric sensitivity analysis
   - Empirical parametric sensitivities
   - Selecting parameters to be estimated
3. Advanced input signal design
   - Sampling time
   - Sufficient exitation

Model types

$$y = \mathcal{M}(x, p)$$

- **linear in parameters**

$$\mathcal{M}(x, p) = p^T \mathcal{F}(x)$$

  where $\mathcal{F}(x)$ is a possibly nonlinear function of the independent variable vector x

- **dynamic**
  discrete time index $k = 0, 1, ..., K, ...$ such that

$$y(k) = \mathcal{M}(x(k), x(k-1), ..., x(k-K); p) \quad , \quad k = K, K+1, ..., n$$

The general parameter estimation problem

Method of parameter estimation: $D^N \to \hat{p}_N$

The general task of parameter estimation:

Given

- measured values : $D[1, N] = D^N = \{(y(k), u(k)) \mid k = 1, \ldots, N\}$
- parametrized nonlinear predictive model :
  $\hat{y}(k|p) = g(k, D[1, k-1]; p)$
  prediction errors: $\varepsilon(k, p) = y(k) - \hat{y}(k|p) \quad , \quad k = 1, \ldots, N$
- norm defined on the prediction error (2-norm, LS):
  $V_{LS}(\theta, D^N) = \frac{1}{N} \sum_{k=1}^{N} (\varepsilon(k, p))^2$

Compute:

The estimated parameter $\hat{p}_N$ *is specified in the time instant* $k = N$ *so that*

$$\hat{p}_{LS} = \hat{p}_{LS}(D^N) = \arg\min_p V_{LS}(p, D^N)$$

Design goal: the **set of parameters to be estimated** and the **strategy for collecting measured data**

Overview – parametric sensitivities

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The notion of parametric sensitivities

For nonlinear parametrized state space models

$$\dot{x}(t) = f(x(t), u(t), p) \quad , \quad x(t_0) = x_0$$
$$y(t) = g(x(t), p) \qquad (*)$$

where $u(t)$ is a known function .

Important

Sensitivity of x and y with respect to p at a given $p = p_0$:

$$S_p^x(t) = \frac{\partial x}{\partial p}(t, p)$$
$$S_p^y(t) = \frac{\partial y}{\partial p}(t, p)$$

The sensitivities depend on the state evolution and on the nominal parameter value, too.

Computation of parametric sensitivities

From (*) a set of ODEs can be analytically derived for the sensitivities

$$\dot{S}_p^x(t) = f_x(x(t), u(t), p) + f_p(t) \ , \ S_p^x(t_0) = \frac{\partial x_0}{\partial p}(p)$$
$$\dot{S}_p^y(t) = g_x(x(t), p) + g_p(t) \quad , \quad S_p^x(t_0) = \frac{\partial g(x_0)}{\partial p}(p) \qquad (**)$$

where f_x is the Jacobian matrix of the nonlinear mapping f with respect to the vector variable x, i.e. $f_x = J^{(f,x)}$.

Important

The two ODEs (*) and (**) can be integrated together, but one needs the analytical computation of the Jacobian matrices $J^{(f,x)}$, $J^{(f,p)}$, $J^{(g,x)}$ and $J^{(g,p)}$.

Empirical computation of parametric sensitivities

Idea: Consider the parameters as disturbances and empirically "observe" the effect of their change on the model output.

Steps for empirical parametric sensitivity analysis

1. Select a steady-state point $\overline{x}$ in the state space by choosing $u(t) \equiv u_C$ $(const)$ and nominal value $p = p_0$
2. Give a unit step disturbance in p of magnitude p_d to the system
3. Observe (record) the output response *till steady-state is reached*

Important

Explore the domain of the model (in both the states and the parameters) by changing

- *the steady-state point* $\overline{x}$
- *the reference parameter value* $p = p_0$
- *the magnitude* $p_d = \pm \pi p$, *where* $0.05 \leq \pi \leq 0.2$

Parameter categories in dynamic models

Model parameters: usually do not depend on time (constants) or change slowly

Important

Categories based on available a priori knowledge:

a. *known physical constants: g, R, etc.*
b. *constants with clear physical meaning : densities, heat capacities, reaction rate coefficients, evaporation heats, boiling/melting points, etc – good quality (high precision) measured/estimated a priori data are available*
c. *empirical constants – a wide validity interval is only known (e.g. the sign or the magnitude)*

Parameters in categories b. and c. are to be estimated

Influential and non-influential parameters

Important

Based on the parametric sensitivity analysis (both theoretical and empirical) the parameters can be categorized into the following classes:

1. *Non-influential : no significant change in the output can be observed even in the case of 20 % parametric perturbation ($\pi = 0.2$) $\Rightarrow$ it cannot be estimated from the output signal*
2. *Influential : a significant but stable response is obtained in the range of perturbance level $0.05 \leq \pi \leq 0.1 \Rightarrow$ it is a good candidate for parameter estimation*
3. *extra sensitive – large unstable response*

Non-influential parameters can be fixed and left out from the set of parameters to be estimated.

Overview – input signal design

Experiment design – a repetition

Aim: to determine the optimal input for parameter estimation

- asymptotic unbiasedness
- minimal variance, uncorrelated elements

Choosing the sampling time

We should aim at

- provide sufficiently high frequency sampling for sufficiently long time,
- the sample should contain enough information for each important and modelled time constant (pole) of the system

Procedure for LTI models:

1. give a unit step to each of the system inputs one-by-one
2. observe the unit step response on each outputs

Approximating nonlinear dynamic models

Nonlinear dynamic models behave approximately in a linear way near a steady-state point (x_0, u_0).

$$\widetilde{y} = J^{(F,x)}\Big|_{x_0,u_0} \cdot \widetilde{x} + J^{(F,u)}\Big|_{x_0,u_0} \cdot \widetilde{u}$$

$$\widetilde{y} = \left(J^{(f,x)}\Big|_0 + J^{(g,x)}\Big|_0 u_0) \right) \cdot \widetilde{x} + g(x_0) \cdot \widetilde{u}$$

Linearized LTI model

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}\widetilde{u}$$

$$\widetilde{y} = \widetilde{C}\widetilde{x} + \widetilde{D}\widetilde{u}$$

$$\widetilde{A} = J^{(f,x)}\Big|_0 + J^{(g,x)}\Big|_0 u_0, \quad \widetilde{B} = g(x_0), \quad \widetilde{C} = J^{(h,x)}\Big|_0, \quad \widetilde{D} = 0$$

Choosing the sampling time for nonlinear dynamic models

Approximately **the procedure for LTI models can be applied** in the neighbourhood of a steady-state point

1. give a unit step to each of the system inputs one-by-one
2. observe the unit step response on each outputs

Based on the unit step responses

- the sampling time should be about 1/4 of (or smaller than) the fastest (smallest) time constant
- the measurement time (number of samples) should be at least 4 times the slowest time constant

Sufficient excitation, test signals

Main considerations:

- *Appropriate signal to noise ratio*
  For this, a suitably chosen test-signal is often added to the normal input of the system to ensure sufficient excitation
- *Asymptotic unbiasedness*
  The inputs should be independent from the other noises and disturbances. Moreover, it is advantageous if the input is (approximately) white noise.

Important

For nonlinear systems the magnitude of the variance of the test signal can be determined experimentally from the unit step responses.